A CONCAVE-CONVEX ELLIPTIC PROBLEM INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract. We study a nonlinear elliptic problem defined in a bounded domain involving fractional powers of the Laplacian operator together with a concave-convex term. We characterize completely the range of parameters for which solutions of the problem exist and prove a multiplicity result.

1. Introduction

In the past decades the problem

$$\begin{cases} -\Delta u &= f(u) \qquad \text{in } \Omega\subset\mathbb{R}^N, \\ \quad u &= 0 \qquad\quad \text{on } \partial\Omega, \end{cases}$$

has been widely investigated. See [3] for a survey, and for example the list (far from complete) [4, 12, 31] for more specific problems, where different nonlinearities and different classes of domains, bounded or not, are considered. Other different diffusion operators, like the p–Laplacian, fully nonlinear operators, etc, have been also treated, see for example [8, 15, 24] and the references there in. We deal here with a nonlocal version of the above problem, for a particular type of nonlinearities, i.e., we study a concave-convex problem involving the fractional Laplacian operator

$$\begin{cases} (-\Delta)^{\alpha/2}u &= \lambda u^q+u^p, \qquad u>0 \qquad \text{in } \Omega, \\ \qquad u &= 0 \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $0<\alpha<2$, $0<q<1<p<\frac{N+\alpha}{N-\alpha}$, $N>\alpha$, $\lambda>0$ and $\Omega\subset\mathbb{R}^N$ a smooth bounded domain.

The nonlocal operator $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^N$ is defined on the Schwartz class through the Fourier transform,

$$[(-\Delta)^{\alpha/2}g]^\wedge(\xi)=(2\pi|\xi|)^\alpha\,\widehat{g}(\xi),$$

or via the Riesz potential, see for example [28, 35] for the precise formula. As usual, $\widehat{g}$ denotes the Fourier Transform of g, $\widehat{g}(\xi)=\int_{\mathbb{R}^N}e^{-2\pi i x\cdot\xi}g(x)\,dx$. Observe that $\alpha=2$ corresponds to the standard local Laplacian.

This type of diffusion operators arises in several areas such as physics, probability and finance, see for instance [6, 7, 20, 39]. In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy process, [7].

There is another way of defining this operator. In fact, in the case $\alpha=1$ there is an explicit form of calculating the half-Laplacian acting on a function u in the whole space $\mathbb{R}^N$, as the normal derivative on the boundary of its harmonic extension to the upper half-space $\mathbb{R}^{N+1}_+$, the so-called Dirichlet to Neumann operator. The "α

derivative" $(-\Delta)^{\alpha/2}$ can be characterized in a similar way, defining the α-harmonic extension to the upper half-space, see [16] and Section 2 for details. This extension is commonly used in the recent literature since it allows to write nonlocal problems in a local way and this permits to use the variational techniques for these kind of problems.

In the case of the operator defined in bounded domains Ω, the above characterization has to be adapted. The fractional powers of a linear positive operator in Ω are defined by means of the spectral decomposition. In [14], the authors consider the fractional operator $(-\Delta)^{1/2}$ defined using the mentioned Dirichlet to Neumann operator, but restricted to the cylinder $\Omega \times \mathbb{R}_+ \subset \mathbb{R}_+^{N+1}$, and show that this definition is coherent with the spectral one, see also [36] for the case $\alpha \neq 1$. We recall that this is not the unique possibility of defining a nonlocal operator related to the fractional Laplacian in a bounded domain. See for instance the definition of the so called *regional fractional Laplacian* in [9, 27], where the authors consider the Riesz integral restricted to the domain Ω. This leads to a different operator related to a Neumann problem.

As to the concave-convex nonlinearity, there is a huge amount of results involving different (local) operators, see for instance [1, 4, 8, 17, 19, 24]. We quoted the work [4] from where some ideas are used in the present paper. In most of the problems considered in those papers a critical exponent appears, which generically separates the range where compactness results can be applied or can not (in the fully nonlinear case the situation is slightly different, but still a critical exponent appears, [17]). In our case, the critical exponent with respect to the corresponding Sobolev embedding is given by $2_\alpha^* = \frac{2N}{N-\alpha}$. This is a reason why problem (1.1) is studied in the subcritical case $p < 2_\alpha^* - 1 = \frac{N+\alpha}{N-\alpha}$; see also the nonexistence result for supercritical nonlinearities in Corollary 4.6.

The main results we prove characterize the existence of solutions of (1.1) in terms of the parameter λ. A competition between the sublinear and superlinear terms plays a role, which leads to different results concerning existence and multiplicity of solutions, among others.

Theorem 1.1. *There exists $\Lambda > 0$ such that for Problem* (1.1) *there holds:*

1. *If $0 < \lambda < \Lambda$ there is a minimal solution. Moreover, the family of minimal solutions is increasing with respect to λ.*
2. *If $\lambda = \Lambda$ there is at least one solution.*
3. *If $\lambda > \Lambda$ there is no solution.*

Moreover, we have the following multiplicity result:

Theorem 1.2. *For each $0 < \lambda < \Lambda$, Problem* (1.1) *has at least two solutions.*

On the contrary, there is a unique solution with small norm.

Theorem 1.3. *For each $0 < \lambda < \Lambda$ fixed, there exists a constant $A > 0$ such that there exists at most one solution u to Problem* (1.1) *verifying*

$$\|u\|_\infty \leq A.$$

For $\alpha \in [1, 2)$ and p subcritical, we also prove that there exists an universal L^∞-bound for every solution independently of λ.

Theorem 1.4. *Let $\alpha \geq 1$. Then there exists a constant $C > 0$ such that, for any $0 < \lambda \leq \Lambda$, every solution to Problem* (1.1) *satisfies*

$$\|u\|_\infty \leq C.$$

The proof of this last result relies on the classical argument of rescaling introduced in [25] which yields to problems on unbounded domains, which require some Liouville-type results, which can be seen in [32]. This is the point where the restriction $\alpha \geq 1$ appears.

The paper is organized as follows: in Section 2 we recollect some properties of the fractional Laplacian in the whole space, and establish a trace inequality corresponding to this operator; the fractional Laplacian in a bounded domain is considered in Section 3, by means of the use of the α-harmonic extension; this includes studying an associated linear equation in the local version. The main section, Section 4, contains the results related to the nonlocal nonlinear problem (1.1), where we prove Theorems 1.1–1.4.

2. The fractional Laplacian in $\mathbb{R}^N$

2.1. **Preliminaries.** Let u be a regular function in $\mathbb{R}^N$. We say that $w = \mathrm{E}_\alpha(u)$ is the α-harmonic extension of u to the upper half-space, $\mathbb{R}^{N+1}_+$, if w is a solution to the problem

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w = u & \text{on } \mathbb{R}^N \times \{y=0\}. \end{cases}$$

In [16] it is proved that

$$\lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y) = -\kappa_\alpha(-\Delta)^{\alpha/2}u(x), \tag{2.1}$$

where $\kappa_\alpha = \frac{2^{1-\alpha}\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)}$. Observe that $\kappa_\alpha = 1$ for $\alpha = 1$ and $\kappa_\alpha \sim 1/(2-\alpha)$ as $\alpha \to 2^-$. As we pointed out in the Introduction, identity (2.1) allows to formulate nonlocal problems involving the fractional powers of the Laplacian in $\mathbb{R}^N$ as local problems in divergence form in the half-space $\mathbb{R}^{N+1}_+$.

The appropriate functional spaces to work with are $X^\alpha(\mathbb{R}^{N+1}_+)$ and $\dot H^{\alpha/2}(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$ and $C_0^\infty(\mathbb{R}^N)$, respectively, under the norms

$$\|\phi\|^2_{X^\alpha} = \int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}|\nabla\phi(x,y)|^2\,dxdy,$$

$$\|\psi\|^2_{\dot H^{\alpha/2}} = \int_{\mathbb{R}^N} |2\pi\xi|^\alpha|\widehat{\psi}(\xi)|^2\,d\xi = \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/4}\psi(x)|^2\,dx.$$

The extension operator is well defined for smooth functions through a Poisson kernel, whose explicit expression is given in [16]. It can also be defined in the space $\dot H^{\alpha/2}(\mathbb{R}^N)$, and in fact

$$\|\mathrm{E}_\alpha(\psi)\|_{X^\alpha} = c_\alpha\|\psi\|_{\dot H^{\alpha/2}}, \quad \forall\,\psi \in \dot H^{\alpha/2}(\mathbb{R}^N), \tag{2.2}$$

where $c_\alpha = \sqrt{\kappa_\alpha}$, see Lemma 2.2. On the other hand, for a function $\phi \in X^\alpha(\mathbb{R}^{N+1}_+)$, we will denote its trace on $\mathbb{R}^N \times \{y = 0\}$ as $\mathrm{Tr}(\phi)$. This trace operator is also well defined and it satisfies

$$\|\,\mathrm{Tr}(\phi)\|_{\dot H^{\alpha/2}} \le c_\alpha^{-1}\|\phi\|_{X^\alpha}. \tag{2.3}$$

2.2. **A trace inequality.** In order to prove regularity of solutions to problem (1.1) we will use a trace immersion. As a first step we show that the corresponding result for the whole space holds. Although most of the results used in order to prove Theorem 2.1 below are known we have collected them for the readers convenience.

First of all we prove inequality (2.3). The Sobolev embedding yields then, that the trace also belongs to $L^{2^*_\alpha}(\mathbb{R}^N)$, where $2^*_\alpha = \frac{2N}{N-\alpha}$. Even the best constant associated to this inclusion is attained and can be characterized.

Theorem 2.1. *For every $z \in X^\alpha(\mathbb{R}^{N+1}_+)$ it holds*

$$\left(\int_{\mathbb{R}^N} |v(x)|^{\frac{2N}{N-\alpha}}\,dx\right)^{\frac{N-\alpha}{N}} \le S(\alpha,N)\int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}|\nabla z(x,y)|^2\,dxdy, \tag{2.4}$$

where $v = \mathrm{Tr}(z)$. The best constant takes the exact value

$$S(\alpha,N) = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{N-\alpha}{2})(\Gamma(N))^{\frac{\alpha}{N}}}{2\pi^{\frac{\alpha}{2}}\Gamma(\frac{2-\alpha}{2})\Gamma(\frac{N+\alpha}{2})(\Gamma(\frac{N}{2}))^{\frac{\alpha}{N}}} \tag{2.5}$$

and it is achieved when v takes the form

$$v(x) = \left(|x-x_0|^2+\tau^2\right)^{-\frac{N+\alpha}{2}}, \tag{2.6}$$

for some $x_0 \in \mathbb{R}^N$, $\tau \in \mathbb{R}$, and $z = \mathrm{E}_\alpha(v)$.

The analogous results for the classical Laplace operator can be found in [22, 31].

The proof of Theorem 2.1 follows from a series of lemmas, that we prove next.

Lemma 2.2. *Let $v \in \dot H^{\alpha/2}(\mathbb{R}^N)$ and let $z = \mathrm{E}_\beta(v)$ be its β-harmonic extension, $\beta \in (\alpha/2, 2)$. Then $z \in X^\alpha(\mathbb{R}^{N+1}_+)$ and moreover there exists a positive universal constant $c(\alpha,\beta)$ such that*

$$\|v\|_{\dot H^{\alpha/2}} = c(\alpha,\beta)\|z\|_{X^\alpha}. \tag{2.7}$$

In particular if $\beta = \alpha$ we have $c(\alpha,\alpha) = 1/\sqrt{\kappa_\alpha}$.

Inequality (2.4) needs only the case $\beta = \alpha$, which was included in the proof of the local characterization of $(-\Delta)^{\alpha/2}$ in [16]. The calculations performed in [16] can be extended to cover the range $\alpha/2 < \beta < 2$ and in particular includes the case $\beta = 1$ proved in [40].

Proof. Since $z = \mathrm{E}_\beta(v)$, by definition z solves $\mathrm{div}(y^{1-\beta}\nabla z) = 0$, which is equivalent to

$$\Delta_x z + \frac{1-\beta}{y}\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$$

Taking Fourier transform in $x \in \mathbb{R}^N$ for $y > 0$ fixed, we have

$$-4\pi^2|\xi|^2\hat z + \frac{1-\beta}{y}\frac{\partial \hat z}{\partial y} + \frac{\partial^2 \hat z}{\partial y^2} = 0.$$

and $\hat{z}(\xi,0)=\hat{v}(\xi)$. Therefore $\hat{z}(\xi,y)=\hat{v}(\xi)\phi_\beta(2\pi|\xi|y)$, where ϕ_β solves the problem

$$-\phi+\frac{1-\beta}{s}\phi'+\phi''=0,\qquad \phi(0)=1,\quad \lim_{s\to\infty}\phi(s)=0. \tag{2.8}$$

In fact, ϕ_β minimizes the functional

$$H_\beta(\phi)=\int_0^\infty(|\phi(s)|^2+|\phi'(s)|^2)s^{1-\beta}\,ds.$$

and it can be shown that it is a combination of Bessel functions, see [29]. More precisely, ϕ_β satisfies the following asymptotic behaviour

$$\begin{aligned}\phi_\beta(s)&\sim 1-c_1s^\beta, &&\text{for } s\to 0,\\ \phi_\beta(s)&\sim c_2 s^{\frac{\beta-1}{2}}e^{-s}, &&\text{for } s\to\infty,\end{aligned} \tag{2.9}$$

where

$$c_1(\beta)=\frac{2^{1-\beta}\Gamma(1-\beta/2)}{\beta\Gamma(\beta/2)},\quad c_2(\beta)=\frac{2^{\frac{1-\beta}{2}}\pi^{1/2}}{\Gamma(\beta/2)}.$$

Now we observe that

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla z(x,y)|^2\,dx &= \int_{\mathbb{R}^N}\left(|\nabla_x z(x,y)|^2+\left|\frac{\partial z}{\partial y}(x,y)\right|^2\right)dx\\ &=\int_{\mathbb{R}^N}\left(4\pi^2|\xi|^2|\hat{z}(\xi,y)|^2+\left|\frac{\partial \hat{z}}{\partial y}(\xi,y)\right|^2\right)d\xi.\end{aligned}$$

Then, multiplying by $y^{1-\alpha}$ and integrating in y,

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}y^{1-\alpha}|\nabla z(x,y)|^2\,dxdy\\ &=\int_0^\infty\int_{\mathbb{R}^N}4\pi^2|\xi|^2|\hat{v}(\xi)|^2\big(|\phi_\beta(2\pi|\xi|y)|^2+|\phi_\beta'(2\pi|\xi|y)|^2\big)y^{1-\alpha}\,d\xi dy\\ &=\int_0^\infty(|\phi_\beta(s)|^2+|\phi_\beta'(s)|^2)s^{1-\alpha}\,ds\int_{\mathbb{R}^N}|2\pi\xi|^\alpha|\hat{v}(\xi)|^2\,d\xi.\end{aligned}$$

Using (2.9) we see that the integral $\int_0^\infty(|\phi_\beta|^2+|\phi_\beta'|^2)s^{1-\alpha}\,ds$ is convergent provided $\beta>\alpha/2$. This proves (2.7) with $c(\alpha,\beta)=(H_\alpha(\phi_\beta))^{-1/2}$. □

Remark 2.1. If $\beta=1$ we have $\phi_1(s)=e^{-s}$, and $H_\alpha(\phi_1)=2^{\alpha-1}\Gamma(2-\alpha)$, see [40]. Moreover, when $\beta=\alpha$, integrating by parts and using the equation in (2.8), and (2.9), we obtain

$$H_\alpha(\phi_\alpha)=\int_0^\infty[\phi_\alpha^2(s)+(\phi_\alpha')^2(s)]s^{1-\alpha}\,ds=-\lim_{s\to0}s^{1-\alpha}\phi_\alpha'(s)=\alpha c_1(\alpha)=\kappa_\alpha. \tag{2.10}$$

Lemma 2.3. *Let $z\in X^\alpha(\mathbb{R}^{N+1}_+)$ and let $w=\mathrm{E}_\alpha(\mathrm{Tr}(z))$ be its α-harmonic associated function (the extension of the trace). Then*

$$\|z\|_{X^\alpha}^2=\|w\|_{X^\alpha}^2+\|z-w\|_{X^\alpha}^2.$$

Proof. Observe that, for $h = z - w$, we have

$$\|z\|_{X^\alpha}^2 = \int_{\mathbb{R}_+^{N+1}} y^{1-\alpha}(|\nabla w|^2 + |\nabla h|^2 + 2\langle \nabla w, \nabla h\rangle).$$

But, since $\mathrm{Tr}(h) = 0$, we have $\displaystyle\int_{\mathbb{R}_+^{N+1}} y^{1-\alpha}\langle \nabla w, \nabla h\rangle\, dxdy = 0.$ □

Lemma 2.4. *If $g \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, and $f \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$, then there exists a constant $\ell(\alpha, N) > 0$ such that*

$$\left|\int f(x)\, g(x)\, dx\right| \le \ell(\alpha, N)\|f\|_{\dot{H}^{\alpha/2}}\|g\|_{\frac{2N}{N+\alpha}}. \tag{2.11}$$

Moreover, the equality in (2.11) *with the best constant holds when f and g take the form* (2.6).

The proof follows and standard argument that can be found, for instance in [21, 40].

Proof. By Parçeval's identity and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N} f(x)\, g(x)\, dx\right)^2 &= \left(\int_{\mathbb{R}^N} \widehat{f}(\xi)\, \widehat{g}(\xi)\, d\xi\right)^2 \\ &\le \left(\int_{\mathbb{R}^N} |2\pi\xi|^\alpha\, |\widehat{f}(\xi)|^2\, d\xi\right)\left(\int_{\mathbb{R}^N} |2\pi\xi|^{-\alpha}\, |\widehat{g}(\xi)|^2\, d\xi\right).\end{aligned}$$

The second term can be written using [30] as

$$\int_{\mathbb{R}^N} |2\pi\xi|^{-\alpha}\, |\widehat{g}(\xi)|^2\, d\xi = b(\alpha, N)\int_{\mathbb{R}^{2N}} \frac{g(x)g(x')}{|x - x'|^{N-\alpha}}\, dxdx',$$

where

$$b(\alpha, N) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{N/2}\Gamma(\frac{\alpha}{2})},$$

We now use the following Hardy-Littlewood-Sobolev inequality,

$$\int_{\mathbb{R}^{2N}} \frac{g(x)g(x')}{|x - x'|^{N-\alpha}}\, dxdx' \le d(\alpha, N)\|g\|_{\frac{2N}{N+\alpha}}^2,$$

see again [30], where

$$d(\alpha, N) = \frac{\pi^{\frac{N-\alpha}{2}}\Gamma(\alpha/2)(\Gamma(N))^{\frac{\alpha}{N}}}{\Gamma((N+\alpha)/2)(\Gamma(N/2))^{\frac{\alpha}{N}}},$$

with equality if g takes the form (2.6). From this we obtain the desired estimate (2.11) with the constant $\ell(\alpha, N) = \sqrt{b(\alpha, N)d(\alpha, N)}$.

When applying Cauchy-Schwarz's inequality, we obtain an identity provided the functions $|\xi|^{\alpha/2}\widehat{f}(\xi)$ and $|\xi|^{-\alpha/2}\widehat{g}(\xi)$ are proportional. This means

$$\widehat{g}(\xi) = c|\xi|^\alpha \widehat{f}(\xi) = c[(-\Delta)^{\alpha/2} f]^\wedge(\xi).$$

We end by observing that if g takes the form (2.6) and $g = c(-\Delta)^{\alpha/2} f$ then f also takes the form (2.6). In fact, the only positive regular solutions to $(-\Delta)^{\alpha/2} f = cf^{\frac{N+\alpha}{N-\alpha}}$ take the form (2.6), see [18]. □

Proof of Theorem 2.1. We apply Lemma 2.4 with $g = |f|^{\frac{N+\alpha}{N-\alpha}-1}f$, then use Lemma 2.2 and conclude using Lemma 2.3. The best constant is $S(\alpha, N) = \ell^2(\alpha, N)/\kappa_\alpha$. □

Remark 2.2. If we let α tend to 2, when $N > 2$, we recover the classical Sobolev inequality for a function in $H^1(\mathbb{R}^N)$, with the same constant, see [38]. In order to pass to the limit in the right-hand side of (2.4), at least formally, we observe that $(2-\alpha)y^{1-\alpha}\,dy$ is a measure on compact sets of $\mathbb{R}_+$ converging (in the weak-* sense) to a Dirac delta. Hence

$$\lim_{\alpha\to 2^-}\int_0^1\left(\int_{\mathbb{R}^N}|\nabla z(x,y)|^2\,dx\right)(2-\alpha)y^{1-\alpha}\,dy = \int_{\mathbb{R}^N}|\nabla v(x)|^2\,dx.$$

We then obtain

$$\left(\int_{\mathbb{R}^N}|v(x)|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}} \le S(N)\int_{\mathbb{R}^N}|\nabla v(x)|^2\,dx,$$

with the best constant $S(N) = \lim_{\alpha\to 2^-}\frac{S(\alpha,N)}{2-\alpha} = \frac{1}{\pi N(N-2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{\frac{N}{2}}$. It is achieved when v takes the form (2.6) with α replaced by 2.

3. The fractional Laplacian in a bounded domain

3.1. **Spectral decomposition.** To define the fractional Laplacian in a bounded domain we follow [14], see also [36]. To this aim we consider the cylinder

$$\mathcal{C}_\Omega = \{(x,y) : x \in \Omega,\ y \in \mathbb{R}_+\} \subset \mathbb{R}^{N+1}_+\,,$$

and denote by $\partial_L\mathcal{C}_\Omega$ its lateral boundary. We also define the energy space

$$X_0^\alpha(\mathcal{C}_\Omega) = \{z \in L^2(\mathcal{C}_\Omega)\,:\, z = 0 \text{ on } \partial_L\mathcal{C}_\Omega,\ \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla z(x,y)|^2\,dxdy < \infty\},$$

with norm

$$\|z\|^2_{X_0^\alpha} := \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla z(x,y)|^2\,dxdy.$$

We want to characterize the space $\Theta(\Omega)$, which is the image of $X_0^\alpha(\mathcal{C}_\Omega)$ under the trace operator,

$$\Theta(\Omega) = \{u = \mathrm{Tr}(w)\,:\, w \in X_0^\alpha(\mathcal{C}_\Omega)\}.$$

In particular we will show that the fractional Laplacian in a bounded domain Ω is well defined for functions in $\Theta(\Omega)$. To this aim, let us start considering the extension operator and fractional Laplacian for smooth functions.

Definition 3.1. *Given a regular function u, we define its α-harmonic extension $w = \mathrm{E}_\alpha(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution to the problem*

$$\left\{\begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla w) &=& 0 & \text{in } \mathcal{C}_\Omega,\\ w &=& 0 & \text{on } \partial_L\mathcal{C}_\Omega,\\ \mathrm{Tr}(w) &=& u & \text{on } \Omega. \end{array}\right. \tag{3.1}$$

As in the whole space, there is also a Poisson formula for the extension operator in a bounded domain, defined through the Laplace transform and the heat semigroup generator $e^{t\Delta}$, see [36] for details.

Definition 3.2. *The fractional operator* $(-\Delta)^{\alpha/2}$ *in* Ω*, acting on a regular function* u*, is defined by*

$$(-\Delta)^{\alpha/2}u(x) = -\frac{1}{\kappa_\alpha}\lim_{y\to0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y), \tag{3.2}$$

where $w = \mathrm{E}_\alpha(u)$ *and* κ_α *is given as in* (2.1).

It is classical that the powers of a positive operator in a bounded domain are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. We show next that in this case this is coherent with the Dirichlet-Neumann operator defined above.

Lemma 3.3. *Let* (φ_j, λ_j) *be the eigenfunctions and eigenvectors of* $-\Delta$ *in* Ω *(with Dirichlet boundary data). Then* $(\varphi_j, \lambda_j^{\alpha/2})$ *are the eigenfunctions and eigenvectors of* $(-\Delta)^{\alpha/2}$ *in* Ω *(with the same boundary conditions). Moreover,* $\mathrm{E}_\alpha(\varphi_j) = \varphi_j(x)\psi(\lambda_j^{1/2}y)$*, where* ψ *solves the problem*

$$\begin{cases} \psi'' + \dfrac{(1-\alpha)}{s}\psi' &= \psi, \qquad s>0,\\ -\displaystyle\lim_{s\to0^+} s^{1-\alpha}\psi'(s) &= \kappa_\alpha,\\ \psi(0) &= 1. \end{cases}$$

The proof of this result, based on separating variables, is straightforward. The function ψ coincides with the solution ϕ_α in problem (2.8).

Lemma 3.4. *Let* $u \in L^2(\Omega)$*. If* $u = \sum a_j\varphi_j$*, where* $\sum a_j^2\lambda_j^{\alpha/2} < \infty$*, then* $\mathrm{E}_\alpha(u) \in X_0^\alpha(\mathcal{C}_\Omega)$ *and*

$$\mathrm{E}_\alpha(u)(x,y) = \sum a_j\varphi_j(x)\psi(\lambda_j^{1/2}y).$$

Proof. The formula for the extension follows immediately from Lemma 3.3.

Put $w = \mathrm{E}_\alpha(u)$. Using the orthogonality of the family $\{\varphi_j\}$, together with $\int_\Omega \varphi_j^2 = 1$, $\int_\Omega |\nabla\varphi_j|^2 = \lambda_j$, and (2.10), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}|\nabla w(x,y)|^2\,dxdy\\
&= \int_0^\infty y^{1-\alpha}\int_\Omega\Big(\sum a_j^2|\nabla\varphi_j(x)|^2\psi(\lambda_j^{1/2}y)^2 + a_j^2\lambda_j\varphi_j(x)^2(\psi'(\lambda_j^{1/2}y))^2\Big)\,dxdy\\
&= \int_0^\infty y^{1-\alpha}\sum a_j^2\lambda_j\Big(\psi(\lambda_j^{1/2}y)^2 + (\psi'(\lambda_j^{1/2}y))^2\Big)\,dy\\
&= \sum a_j^2\lambda_j^{\alpha/2}\int_0^\infty s^{1-\alpha}\Big(\psi(s)^2 + (\psi'(s))^2\Big)\,ds = \kappa_\alpha\sum a_j^2\lambda_j^{\alpha/2} < \infty.
\end{aligned}$$

□

As in Section 2, we get that the extension operator minimizes the norm in $X_0^\alpha(\mathcal{C}_\Omega)$.

Lemma 3.5. *Let* $z \in X_0^\alpha(\mathcal{C}_\Omega)$ *and let* $w = \mathrm{E}_\alpha(\mathrm{Tr}(z))$*. Then*

$$\|z\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + \|z-w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}.$$

Lemmas 3.4 and 3.5 imply that the space $\Theta(\Omega)$ coincides with the space

$$H_0^{\alpha/2}(\Omega) = \left\{ u = \sum a_j \varphi_j \in L^2(\Omega) \;:\; \sum a_j^2 \lambda_j^{\alpha/2} < \infty \right\},$$

equipped with the norm

$$\|u\|_{H_0^{\alpha/2}(\Omega)} = \left(\sum a_j^2 \lambda_j^{\alpha/2} \right)^{1/2} = \|(-\Delta)^{\alpha/4} u\|_2.$$

Observe that Lemma 3.4 gives

$$\|u\|_{H_0^{\alpha/2}(\Omega)} = \kappa_\alpha^{-1/2} \| \operatorname{E}_\alpha(u)\|_{X_0^\alpha(\mathcal{C}_\Omega)},$$

which is the same as in (2.2) but for the spaces and norms involved. As a direct consequence we get that the fractional Laplacian is well defined in this space.

Corollary 3.6. *Let $u \in H_0^{\alpha/2}(\Omega)$, then $(-\Delta)^{\alpha/2} u = \sum a_j \lambda_j^{\alpha/2} \varphi_j$.*

Let now $v \in X_0^\alpha(\mathcal{C}_\Omega)$. Its extension by zero outside the cylinder $\mathcal{C}_\Omega$ can be approximated by functions with compact support in $\overline{\mathbb{R}^{N+1}_+}$. Thus, the trace inequality (2.4), together with Hölder's inequality, gives a trace inequality for bounded domains.

Corollary 3.7. *For any $1 \le r \le \frac{2N}{N-\alpha}$, and every $z \in X_0^\alpha(\mathcal{C}_\Omega)$, it holds*

$$\left(\int_\Omega |v(x)|^r \, dx \right)^{2/r} \le C(r, \alpha, N, |\Omega|) \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla z(x,y)|^2 \, dxdy, \tag{3.3}$$

where $v = \operatorname{Tr}(z)$.

3.2. **The linear problem.** We now use the extension problem (3.1) and the expression (3.2) to reformulate the nonlocal problems in a local way. Let g be a regular function and consider the following problems: the nonlocal problem

$$\left\{ \begin{array}{rcll} (-\Delta)^{\alpha/2} u & = & g(x) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3.4}$$

and the corresponding local one

$$\left\{ \begin{array}{rcll} -\operatorname{div}(y^{1-\alpha} \nabla w) & = & 0 & \text{in } \mathcal{C}_\Omega, \\ w & = & 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ -\dfrac{1}{\kappa_\alpha} \lim\limits_{y \to 0^+} y^{1-\alpha} \dfrac{\partial w}{\partial y} & = & g(x) & \text{on } \Omega. \end{array} \right. \tag{3.5}$$

We want to define the concept of solution to (3.4), which is done in terms of the solution to problem (3.5).

Definition 3.8. *We say that $w \in X_0^\alpha(\mathcal{C}_\Omega)$ is an energy solution to problem* (3.5), *if for every function $\varphi \in \mathcal{C}_0^1(\mathcal{C}_\Omega)$ it holds*

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha} \big\langle \nabla w(x,y), \nabla \varphi(x,y) \big\rangle \, dxdy = \int_\Omega \kappa_\alpha g(x) \varphi(x,0) \, dx. \tag{3.6}$$

In fact more general test functions can be used in the above formula, whenever the integrals make sense. A supersolution (subsolution) is a function that verifies (3.6) with equality replaced by $\ge$ ($\le$) for every nonnegative test function.

Definition 3.9. *We say that* $u \in H_0^{\alpha/2}(\Omega)$ *is an energy solution to problem* (3.4) *if* $u = \mathrm{Tr}(w)$ *and* w *is an energy solution to problem* (3.5).

In order to deal with problem (3.5) we will assume, without loss of generality, $\kappa_\alpha = 1$, by changing the function g.

In [13] this linear problem is also mentioned. There some results are obtained using the theory of degenerate elliptic equations developed in [23], in particular a regularity result for bounded solutions to this problem is obtained in [13]. We prove here that the solutions are in fact bounded if g satisfies a minimal integrability condition.

Theorem 3.10. *Let* w *be a solution to problem* (3.5)*. If* $g \in L^r(\Omega)$*,* $r > \frac{N}{\alpha}$*, then* $w \in L^\infty(\mathcal{C}_\Omega)$*.*

Proof. The proof follows the same ideas as in [26, Theorem 8.15] and uses the trace inequality (3.3). Without loss of generality we may assume $w \geq 0$, and this simplifies notation. The general case is obtained in a similar way.

We define for $\beta \geq 1$ and $K \geq k$ (k to be chosen later) a $\mathcal{C}^1([k,\infty))$ function H, as follows:

$$H(z) = \begin{cases} z^\beta - k^\beta, & z \in [k, K], \\ \text{linear}, & z > K. \end{cases}$$

Let us also define $v = w + k$ and choose as test function φ,

$$\varphi = G(v) = \int_k^v |H'(s)|^2\, ds, \qquad \nabla\varphi = |H(v)|^2 \nabla v.$$

Replacing this test function into the definition of energy solution we obtain on one hand:

$$\begin{aligned} \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla w, \nabla \varphi \rangle\, dxdy &= \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla v|^2 |H'(v)|^2\, dxdy \\ &= \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla H(v)|^2\, dxdy \\ &\geq \left(\int_\Omega |H(v)|^{\frac{2N}{N-\alpha}}\, dx \right)^{\frac{N-\alpha}{N}} = \|H(v)\|^2_{\frac{2N}{N-\alpha}}, \end{aligned} \tag{3.7}$$

where the last inequality follows by (3.3).

On the other hand

$$\begin{aligned} \int_\Omega g(x)\varphi(x,0)\, dx &= \int_\Omega g(x) G(v)\, dx \leq \int_\Omega g(x) v G'(v)\, dx \\ &\leq \frac{1}{k} \int_\Omega g(x) v^2 |H'(v)|^2\, dx = \frac{1}{k} \int_\Omega g(x) |vH'(v)|^2\, dx. \end{aligned} \tag{3.8}$$

Inequality (3.7) together with (3.8), leads to

$$\|H(v)\|_{\frac{2N}{N-\alpha}} \leq \left(\frac{1}{k} \|g\|_r \right)^{1/2} \|(vH'(v))^2\|^{1/2}_{\frac{r}{r-1}} = \|vH'(v)\|_{\frac{2r}{r-1}}, \tag{3.9}$$

by choosing $k = \|g\|_r$. Letting $K \to \infty$ in the definition of H, the inequality (3.9) becomes

$$\|v\|_{\frac{2N\beta}{N-\alpha}} \leq \|v\|_{\frac{2r\beta}{r-1}}.$$

Hence for all $\beta \geq 1$ the inclusion $v \in L^{\frac{2r\beta}{r-1}}(\Omega)$ implies the stronger inclusion $v \in L^{\frac{2N\beta}{N-\alpha}}(\Omega)$, since $\frac{2N\beta}{N-\alpha} > \frac{2r\beta}{r-1}$ provided $r > \frac{N}{\alpha}$. The result follows now, as in [26], by an iteration argument, starting with $\beta = \frac{N(r-1)}{r(N-\alpha)} > 1$ and $v \in L^{\frac{2N}{N-\alpha}}(\Omega)$. This gives $v \in L^\infty(\Omega)$, and then $w \in L^\infty(\mathcal{C}_\Omega)$. In fact we get the estimate

$$\|w\|_\infty \leq c(\|w\|_{X^\alpha} + \|g\|_r).$$

□

Corollary 3.11. *Let w be a solution to problem* (3.5). *If $g \in L^\infty(\Omega)$, then $w \in \mathcal{C}^\gamma(\overline{\mathcal{C}_\Omega})$ for some $\gamma \in (0,1)$.*

Proof. It follows directly from Theorem 3.10 and [13, Lemma 4.4]. □

4. The nonlinear nonlocal problem

4.1. **The local realization.** We deal now with the core of the paper; i.e. the study of the nonlocal problem (1.1). We write that problem in local version in the following way: a solution to problem (1.1) is a function $u = \mathrm{Tr}(w)$, the trace of w on $\Omega \times \{y = 0\}$, where w solves the local problem

$$\left\{\begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla w) & = & 0 & \text{in } \mathcal{C}_\Omega, \\ w & = & 0 & \text{on } \partial_L\mathcal{C}_\Omega, \\ \dfrac{\partial w}{\partial \nu^\alpha} & = & f(w) & \text{in } \Omega, \end{array}\right. \tag{4.1}$$

where

$$\frac{\partial w}{\partial \nu^\alpha}(x) = -\lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y). \tag{4.2}$$

In order to simplify notation in what follows we will denote w for the function defined in the cylinder $\mathcal{C}_\Omega$ as well as for its trace $\mathrm{Tr}(w)$ on $\Omega \times \{y = 0\}$.

As we have said, we will focus on the particular nonlinearity

$$f(s) = f_\lambda(s) = \lambda s^q + s^p. \tag{4.3}$$

However many auxiliary results will be proved for more general reactions f satisfying the growth condition

$$0 \leq f(s) \leq c(1 + |s|^p), \qquad \text{for some } p > 0. \tag{4.4}$$

Remark 4.1. In the definition (4.2) we have neglected the constant κ_α appearing in (3.2) by a simple rescaling. Therefore, the results on the coefficient λ for the local problem (4.1)–(4.3) in this section are translated into problem (1.1) with λ multiplied by $\kappa_\alpha^{p(q-1)-1}$.

Following Definition 3.8, we say that $w \in X_0^\alpha(\mathcal{C}_\Omega)$ is an energy solution of (4.1) if the following identity holds

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w, \nabla \varphi\rangle\, dxdy = \int_\Omega f(w)\varphi\, dx$$

for every regular test function φ. In the analogous way we define sub- and supersolution.

We consider now the functional

$$J(w) = \frac{1}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy - \int_\Omega F(w)\,dx,$$

where $F(s) = \int_0^s f(\tau)\,d\tau$. For simplicity of notation, we define $f(s) = 0$ for $s \le 0$. Recall that the trace satisfies $w \in L^r(\Omega)$, for every $1 \le r \le \frac{2N}{N-\alpha}$ if $N > \alpha$, $1 < r \le \infty$ if $N \le \alpha$. In particular if $p \le \frac{N+\alpha}{N-\alpha}$, and f verifies (4.4) then $F(w) \in L^1(\Omega)$, and the functional is well defined and bounded from below.

It is well known that critical points of J are solutions to (4.1) with a general reaction f. We consider also the minimization problem

$$I = \inf\Big\{\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy \,:\, w \in X_0^\alpha(C_\Omega),\, \int_\Omega F(w)\,dx = 1\Big\},$$

for which, by classical variational techniques, one has that below the critical exponent, the infimum I is achieved.

Proposition 4.1. *If f satisfies* (4.4) *with $p < \frac{N+\alpha}{N-\alpha}$, then there exists a nonnegative function $w \in X_0^\alpha(\mathcal{C}_\Omega)$ for which I is achieved.*

We now establish two preliminary results. The first one is a classical procedure of sub- and supersolutions to obtain a solution. We omit its proof.

Lemma 4.2. *Assume there exist a subsolution w_1 and a supersolution w_2 to problem* (4.1) *verifying $w_1 \le w_2$. Then there also exists a solution w satisfying $w_1 \le w \le w_2$ in $\mathcal{C}_\Omega$.*

The second one is a comparison result for concave nonlinearities. The proof follows the lines of the corresponding one for the Laplacian performed in [10].

Lemma 4.3. *Assume that $f(t)$ is a function verifying that $f(t)/t$ is decreasing for $t > 0$. Consider $w_1, w_2 \in X_0^\alpha(\mathcal{C}_\Omega)$ positive subsolution, supersolution respectively to problem* (4.1)*. Then $w_1 \le w_2$ in $\overline{\mathcal{C}_\Omega}$.*

Proof. By definition we have, for the nonnegative test functions φ_1 and φ_2 to be chosen in an appropriate way,

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha}\big\langle\nabla w_1, \nabla\varphi_1\big\rangle\,dxdy \le \int_\Omega f(w_1)\varphi_1\,dx,$$
$$\int_{\mathcal{C}_\Omega} y^{1-\alpha}\big\langle\nabla w_2, \nabla\varphi_2\big\rangle\,dxdy \ge \int_\Omega f(w_2)\varphi_2\,dx.$$

Now let $\theta(t)$ be a smooth nondecreasing function such that $\theta(t) = 0$ for $t \le 0$, $\theta(t) = 1$ for $t \ge 1$, and set $\theta_\varepsilon(t) = \theta(t/\varepsilon)$. If we put, in the above inequalities

$$\varphi_1 = w_2\,\theta_\varepsilon(w_1 - w_2), \qquad \varphi_2 = w_1\,\theta_\varepsilon(w_1 - w_2),$$

we get

$$I_1 \ge \int_\Omega w_1 w_2\Big(\frac{f(w_2)}{w_2} - \frac{f(w_1)}{w_1}\Big)\theta_\varepsilon(w_1 - w_2)\,dx,$$

where

$$I_1 := \int_{\mathcal{C}_\Omega} y^{1-\alpha} \big\langle w_1 \nabla w_2 - w_2 \nabla w_1, \nabla(w_1 - w_2)\big\rangle\, \theta'_\varepsilon(w_1 - w_2)\, dxdy.$$

Now we estimate I_1 as follows:

$$\begin{aligned} I_1 &\le \int_{\mathcal{C}_\Omega} y^{1-\alpha} \big\langle \nabla w_1, (w_1 - w_2)\nabla(w_1 - w_2)\big\rangle\, \theta'_\varepsilon(w_1 - w_2)\, dxdy \\ &= \int_{\mathcal{C}_\Omega} y^{1-\alpha} \big\langle \nabla w_1, \nabla \gamma_\varepsilon(w_1 - w_2)\big\rangle\, dxdy \end{aligned}$$

where $\gamma'_\epsilon(t) = t\theta'_\varepsilon(t)$. Therefore, since $0 \le \gamma_\varepsilon \le \varepsilon$, we have

$$I_1 \le \int_\Omega f(w_1)\gamma_\varepsilon(w_1 - w_2)\, dx \le c\varepsilon.$$

We end as in [4]. Letting ε tend to zero, we obtain

$$\int_{\Omega \cap \{w_1 > w_2\}} w_1 w_2 \left(\frac{f(w_2)}{w_2} - \frac{f(w_1)}{w_1} \right) dx \le 0,$$

which together with the hypothesis on f gives $w_1 \le w_2$ in Ω. Comparison in $\mathcal{C}_\Omega$ follows easily by the maximum principle. □

Now we show that the solutions to problem (4.1)–(4.4) are bounded and Hölder continuous. Later on, in Section 4.4, we will obtain a uniform L^∞-estimate in the case where f is given by (4.3) and the convex power is subcritical.

Proposition 4.4. *Let f satisfy (4.4) with $p < \frac{N+\alpha}{N-\alpha}$, and let $w \in X_0^\alpha(\mathcal{C}_\Omega)$ be an energy solution to problem (4.1). Then $w \in L^\infty(\mathcal{C}_\Omega) \cap \mathcal{C}^\gamma(\Omega)$ for some $0 < \gamma < 1$.*

Proof. The proof follows closely the technique of [11]. As in the proof of Theorem 3.10, we assume $w \ge 0$. We consider, formally, the test function $\varphi = w^{\beta - p}$, for some $\beta > p + 1$. The justification of the following calculations can be made substituting φ by some approximated truncature. We therefore proceed with the formal analysis. We get, using the trace immersion, the inequality

$$\left(\int_\Omega w^{\frac{(\beta - p + 1)N}{N-\alpha}} \right)^{\frac{N-\alpha}{N}} \le C(\beta, \alpha, N, \Omega) \int_\Omega w^\beta.$$

This estimate allows to obtain the following iterative process

$$\|w\|_{\beta_{j+1}} \le C \|w\|_{\beta_j}^{\frac{\beta_j}{\beta_j - p + 1}},$$

with $\beta_{j+1} = \frac{N}{N-\alpha}(\beta_j + 1 - p)$. To have $\beta_{j+1} > \beta_j$ we need $\beta_j > \frac{(p-1)N}{\alpha}$. Since $w \in L^{2^*_\alpha}(\Omega)$, starting with $\beta_0 = \frac{2N}{N-\alpha}$, we get the above restriction provided $p < \frac{N+\alpha}{N-\alpha}$. It is clear that in a finite number of steps we get, for $g(x) = f(w(x,0))$, the regularity $g \in L^r$ for some $r > \frac{N}{\alpha}$. As a consequence, we obtain the conclusion applying Theorem 3.10 and Corollary 3.11. □

4.2. **A nonexistence result.**

Theorem 4.5. *Assume f is a C^1 function with primitive F, and w is an energy solution to problem (4.1). Then the following Pohozaev-type identity holds*
$$\frac{1}{2}\int_{\partial_L\mathcal{C}_\Omega} y^{1-\alpha}\big\langle (x,y),\nu\big\rangle |\nabla w|^2\,d\sigma - N\int_\Omega F(w)\,dx + \frac{N-\alpha}{2}\int_\Omega wf(w)\,dx = 0.$$

Proof. Just use the identity
$$\begin{aligned}&\big\langle (x,y),\nu\big\rangle y^{\alpha-1}\operatorname{div}(y^{1-\alpha}\nabla w) + \operatorname{div}\Big[y^{1-\alpha}\Big(\big\langle (x,y),\nabla w\big\rangle - \frac{1}{2}(x,y)|\nabla w|^2\Big)\Big]\\ &+\Big(\frac{N+2-\alpha}{2}-1\Big)|\nabla w|^2 = 0,\end{aligned}$$
where ν is the (exterior) normal vector to $\partial\Omega$. It is calculus matter to check this equality. □

Corollary 4.6. *If Ω is starshaped and the nonlinearity f satisfies the inequality $((N-\alpha)sf(s) - 2NF(s)) \ge 0$, then problem (4.1) has no solution. In particular, in the case $f(s) = s^p$ this means that there is no solution for any $p \ge \frac{N+\alpha}{N-\alpha}$.*

The case $\alpha = 1$ has been proved in [14]. The corresponding result for the Laplacian (problem (1.1) with $\alpha = 2$) comes from [34].

4.3. **Proof of Theorems 1.1-1.2.** We prove here Theorems 1.1-1.2 in terms of the solution of the local version (4.1). For the sake of readability we split the proof of Theorem 1.1 into several lemmas. From now on we will denote
$$(P_\lambda) \equiv \left\{\begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla w) &=& 0 & \text{in } \ \mathcal{C}_\Omega,\\ w &=& 0 & \text{on } \ \partial_L\mathcal{C}_\Omega,\\ \dfrac{\partial w}{\partial \nu^\alpha} &=& \lambda w^q + w^p, \quad w>0 & \text{in } \ \Omega,\end{array}\right.$$
and consider the associated energy functional
$$J_\lambda(w) = \frac{1}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy - \int_\Omega F_\lambda(w)\,dx,$$
where
$$F_\lambda(s) = \frac{\lambda}{q+1}s^{q+1} + \frac{1}{p+1}s^{p+1}.$$

Lemma 4.7. *Let Λ be defined by*
$$\Lambda = \sup\{\lambda > 0 \ : \ \textit{Problem } (P_\lambda) \textit{ has solution}\}.$$
Then $\Lambda < \infty$.

Proof. Consider the eigenvalue problem associated to the first eigenvalue λ_1, and let $\varphi_1 > 0$ be an associated eigenfunction, i.e.,
$$\left\{\begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla \varphi_1) &=& 0 & \text{in } \ \mathcal{C}_\Omega,\\ \varphi_1 &=& 0 & \text{on } \ \partial_L\mathcal{C}_\Omega,\\ \dfrac{\partial \varphi_1}{\partial \nu^\alpha} &=& \lambda_1\varphi_1 & \text{in } \ \Omega.\end{array}\right.$$

Then using φ_1 as a test function in (P_λ) we have that

$$\int_\Omega (\lambda w^q + w^p)\varphi_1\, dx = \lambda_1 \int_\Omega w\varphi_1\, dx. \tag{4.5}$$

There exist positive constants c, δ such that $\lambda t^q + t^p > c\lambda^\delta t$, for any $t > 0$. Since $u > 0$ we obtain, using (4.5), that $c\lambda^\delta < \lambda_1$ which implies $\Lambda < \infty$. $\square$

This proves the third statement in Theorem 1.1.

Lemma 4.8. *Problem (P_λ) has a positive solution for every $0 < \lambda < \Lambda$. Moreover, the family $\{w_\lambda\}$ of minimal solutions is increasing with respect to λ.*

Remark 4.2. Although this Λ is not exactly the same as that of Theorem 1.1, see Remark 4.1, we have not changed the notation for the sake of simplicity.

Proof. The associated functional to problem (P_λ) verifies, using Corollary 3.7,

$$\begin{array}{rcl} J_\lambda(w) & = & \displaystyle\frac{1}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy - \int_\Omega F_\lambda(w)\, dx \\ & \geq & \displaystyle\frac{1}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy - \lambda C_1 \left(\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy\right)^{\frac{q+1}{2}} \\ & & \displaystyle -C_2 \left(\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy\right)^{\frac{p+1}{2}}, \end{array}$$

for some positive constants C_1 and C_2. Then for λ small enough there exist two solutions of problem (P_λ), one given by minimization and another one given by the Mountain-Pass Theorem, [5]. The proof is standard, based on the geometry of the function $g(t) = \frac{1}{2}t^2 - \lambda C_1 t^{q+1} - C_2 t^{p+1}$, see for instance [24] for more details.

We now show that there exists a solution for every $\lambda \in (0, \Lambda)$. By definition of Λ, we know that there exists a solution corresponding to any value of λ close to Λ. Let us denote it by μ, and let w_μ be the associated solution. Now w_μ is a supersolution for all problems (P_λ) with $\lambda < \mu$. Take v_λ the unique solution to problem (4.1) with $f(s) = \lambda s^q$. Obviously v_λ is a subsolution to problem (P_λ). By Lemma 4.3 $v_\lambda \leq w_\mu$. Therefore by Lemma 4.2 we conclude that there is a solution for all $\lambda \in (0, \mu)$, and as a consequence, for the whole open interval $(0, \Lambda)$. Moreover, this solution is the minimal one. The monotonicity follows directly from the comparison lemma. $\square$

This proves the first statement in Theorem 1.1.

Lemma 4.9. *Problem (P_λ) has at least one solution if $\lambda = \Lambda$.*

Proof. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \nearrow \Lambda$. We denote by $w_n = w_{\lambda_n}$ the minimal solution to problem (P_{λ_n}). As in [4], we can prove that the linearized equation at the minimal solution has nonnegative eigenvalues. Then it follows, as in [4] again, $J_{\lambda_n}(w_n) < 0$. Since $J'(w_\lambda) = 0$, one easily gets the bound $\|w_n\|_{X_0^\alpha(\mathcal{C}_\Omega)} \leq k$. Hence, there exists a weakly convergent subsequence in $X_0^\alpha(\mathcal{C}_\Omega)$ and as a consequence a weak solution of (P_λ) for $\lambda = \Lambda$. $\square$

This proves the second statement in Theorem 1.1 and finishes the proof of this theorem.

Proof of Theorem 1.2. In order to find a second solution we follow some arguments in [4]. It is essential to have that the first solution is given as a local minimum of the associated functional, J_λ. To prove this last assertion we modify some arguments developed in [2].

Let $\lambda_0 \in (0, \Lambda)$ be fixed and consider $\lambda_0 < \bar{\lambda}_1 < \Lambda$. Take $\phi_0 = w_{\lambda_0}$, $\phi_1 = w_{\bar{\lambda}_1}$ the two minimal solutions to problem (P_λ) with $\lambda = \lambda_0$ and $\lambda = \bar{\lambda}_1$ respectively, then by comparison, $\phi_0 < \phi_1$. We define

$$M = \{w \in X_0^\alpha(\mathcal{C}_\Omega) : \, 0 \le w \le \phi_1\}.$$

Notice that M is a convex closed set of $X_0^\alpha(\mathcal{C}_\Omega)$. Since J_{λ_0} is bounded from bellow in M and it is semicontinuous on M, we get the existence of $\underline{\omega} \in M$ such that $J_{\lambda_0}(\underline{\omega}) = \inf_{w\in M} J_{\lambda_0}(w)$. Let v_0 be the unique positive solution to problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla v_0) & = & 0 & \text{in } \ \mathcal{C}_\Omega, \\ v_0 & = & 0 & \text{on } \ \partial_L\mathcal{C}_\Omega, \\ \dfrac{\partial v_0}{\partial \nu^\alpha} & = & v_0^q & \text{in } \ \Omega. \end{array} \right. \tag{4.6}$$

Since for $0 < \varepsilon << \lambda_0$, and $J_{\lambda_0}(\varepsilon v_0) < 0$, we have $\varepsilon v_0 \in M$, then $\underline{\omega} \ne 0$. Therefore $J_{\lambda_0}(\underline{\omega}) < 0$. By arguments similar to those in [37, Theorem 2.4], we obtain that $\underline{\omega}$ is a solution to problem (P_{λ_0}). There are two possibilities:

- If $\underline{\omega} \not\equiv w_{\lambda_0}$, then the result follows.
- If $\underline{\omega} \equiv w_{\lambda_0}$, we have just to prove that $\underline{\omega}$ is a local minimum of J_{λ_0}. Then the conclusion follows by using an argument close to the one in [4], so we omit the complete details.

We argue by contradiction.

Suppose that $\underline{\omega}$ is not a local minimum of J_{λ_0} in $X_0^\alpha(\mathcal{C}_\Omega)$, then there exists a sequence $\{v_n\} \subset X_0^\alpha(\mathcal{C}_\Omega)$ such that $\|v_n - \underline{\omega}\|_{X_0^\alpha} \to 0$ and $J_{\lambda_0}(v_n) < J_{\lambda_0}(\underline{\omega})$.

Let $w_n = (v_n - \phi_1)^+$ and $z_n = \max\{0, \min\{v_n, \phi_1\}\}$. It is clear that $z_n \in M$ and

$$z_n(x,y) = \left\{ \begin{array}{ll} 0 & \text{if } v_n(x,y) \le 0, \\ v_n(x,y) & \text{if } 0 \le v_n(x,y) \le \phi_1(x,y), \\ \phi_1(x,y) & \text{if } \phi_1(x,y) \le v_n(x,y). \end{array} \right.$$

We set

$$\begin{array}{ll} T_n \equiv \{(x,y) \in \mathcal{C}_\Omega : \, z_n(x,y) = v_n(x,y)\}, & S_n \equiv \operatorname{supp}(w_n), \\ \widetilde{T}_n = \overline{T_n} \cap \Omega, & \widetilde{S}_n = S_n \cap \Omega. \end{array}$$

Notice that $\operatorname{supp}(v_n^+) = T_n \cup S_n$. We claim that

$$|\widetilde{S}_n|_\Omega \to 0 \quad \text{as } n \to \infty, \tag{4.7}$$

where $|A|_\Omega \equiv \int_\Omega \chi_A(x)\, dx$.

By the definition of F_λ, we set $F_{\lambda_0}(s) = \dfrac{\lambda_0}{q+1} s_+^{q+1} + \dfrac{1}{p+1} s_+^{p+1}$, for $s \in \mathbb{R}$, and get

$$\begin{aligned}
J_{\lambda_0}(v_n) =& \frac{1}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla v_n|^2\,dxdy - \int_\Omega F_{\lambda_0}(v_n)\,dx \\
=& \frac{1}{2}\int_{T_n} y^{1-\alpha}|\nabla z_n|^2\,dxdy - \int_{\widetilde{T}_n} F_{\lambda_0}(z_n)\,dx + \frac{1}{2}\int_{S_n} y^{1-\alpha}|\nabla v_n|^2\,dxdy \\
&- \int_{\widetilde{S}_n} F_{\lambda_0}(v_n)\,dx + \frac{1}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla v_n^-|^2\,dxdy \\
=& \frac{1}{2}\int_{T_n} y^{1-\alpha}|\nabla z_n|^2\,dxdy - \int_{\widetilde{T}_n} F_{\lambda_0}(z_n)\,dx \\
&+ \frac{1}{2}\int_{S_n} y^{1-\alpha}|\nabla (w_n+\phi_1)|^2\,dxdy - \int_{\widetilde{S}_n} F_{\lambda_0}(w_n+\phi_1)\,dx \\
&+ \frac{1}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla v_n^-|^2\,dxdy.
\end{aligned}$$

Since

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla z_n|^2\,dxdy = \int_{T_n} y^{1-\alpha}|\nabla v_n|^2\,dxdy + \int_{S_n} y^{1-\alpha}|\nabla \phi_1|^2\,dxdy$$

and

$$\int_\Omega F_{\lambda_0}(z_n)\,dx = \int_{\widetilde{T}_n} F_{\lambda_0}(v_n)\,dx + \int_{\widetilde{S}_n} F_{\lambda_0}(\phi_1)\,dx,$$

by using the fact that ϕ_1 is a supersolution to (P_λ) with $\lambda = \lambda_0$, we conclude that

$$\begin{aligned}
J_{\lambda_0}(v_n) \quad &= \quad J_{\lambda_0}(z_n) + \frac{1}{2}\int_{S_n} y^{1-\alpha}(|\nabla(w_n+\phi_1)|^2 - |\nabla\phi_1|^2)\,dxdy \\
&\quad\quad - \int_{\widetilde{S}_n} (F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1))\,dx + \frac{1}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla v_n^-|^2\,dxdy \\
&\geq \quad J_{\lambda_0}(z_n) + \frac{1}{2}\|w_n\|^2_{X_0^\alpha} + \frac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\
&\quad\quad - \int_\Omega \{F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1) - (F_{\lambda_0})_u(\phi_1)w_n\}\,dx \\
&\geq \quad J_{\lambda_0}(\underline{\omega}) + \frac{1}{2}\|w_n\|^2_{X_0^\alpha} + \frac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\
&\quad\quad - \int_\Omega \{F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1) - (F_{\lambda_0})_u(\phi_1)w_n\}\,dx.
\end{aligned}$$

On one hand, taking into account that $0 < q+1 < 2$, one obtains that

$$0 \leq \frac{1}{q+1}(w_n+\phi_1)^{q+1} - \frac{1}{q+1}\phi_1^{q+1} - \phi_1^q w_n \leq \frac{q}{2}\frac{w_n^2}{\phi_1^{1-q}}.$$

The well known Picone's inequality (see [33]) establish:

$$|\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\cdot\nabla v \geq 0,$$

for differentiable functions $v > 0$, $u \geq 0$. In our case, by an approximation argument we get

$$\lambda_0 \int_\Omega \frac{w_n^2}{\phi_1^{1-q}}dx \leq \|w_n\|^2_{X_0^\alpha}.$$

On the other hand, since $p+1>2$,

$$\begin{aligned} 0 &\leq \frac{1}{p+1}(w_n+\phi_1)^{p+1} - \frac{1}{p+1}\phi_1^{p+1} - \phi_1^p w_n \leq \frac{r}{2} w_n^2 (w_n+\phi_1)^{p-1} \\ &\leq C(p)(\phi_1^{p-1} w_n^2 + w_n^{p+1}). \end{aligned}$$

Hence using that $p+1<2^*_\alpha$ and the claim (4.7)

$$\int_\Omega \left\{ \frac{1}{p+1}(w_n+\phi_1)^{p+1} - \frac{1}{p+1}\phi_1^{p+1} - \phi_1^p w_n \right\} dx \leq o(1)\|w_n\|^2_{X_0^\alpha}.$$

As a consequence we obtain that

$$\begin{aligned} J_{\lambda_0}(v_n) &\geq J_{\lambda_0}(\underline{\omega}) + \frac{1}{2}\|w_n\|^2_{X_0^\alpha}(1-q-o(1)) + \frac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\ &\equiv J_{\lambda_0}(\underline{\omega}) + \frac{1}{2}\|w_n\|^2_{X_0^\alpha}(1-q-o(1)) + o(1). \end{aligned}$$

Since $q<1$, there results that $J_{\lambda_0}(\underline{\omega}) > J_{\lambda_0}(v_n) \geq J_{\lambda_0}(\underline{\omega})$ for $n>n_0$, a contradiction with the main hypothesis. Hence $\underline{\omega}$ is a minimum.

To finish the proof we have to prove the claim (4.7). For $\varepsilon>0$ small, and $\delta>0$ (δ to be chosen later), we consider

$$\begin{aligned} E_n &= \{x\in\Omega:\ v_n(x)\geq\phi_1(x) \quad\wedge\quad \phi_1(x)>\underline{\omega}(x)+\delta\}, \\ F_n &= \{x\in\Omega:\ v_n(x)\geq\phi_1(x) \quad\wedge\quad \phi_1(x)\leq\underline{\omega}(x)+\delta\}. \end{aligned}$$

Using the fact that

$$\begin{aligned} 0 &= |\{x\in\Omega:\ \phi_1(x)<\underline{\omega}(x)\}| = \left|\bigcap_{j=1}^{\infty}\left\{x\in\Omega:\ \phi_1(x)\leq\underline{\omega}(x)+\frac{1}{j}\right\}\right| \\ &= \lim_{j\to\infty}\left|\left\{x\in\Omega:\ \phi_1(x)\leq\underline{\omega}(x)+\frac{1}{j}\right\}\right|, \end{aligned}$$

we get for j_0 large enough, that if $\delta<\frac{1}{j_0}$ then

$$|\{x\in\Omega:\ \phi_1(x)\leq\underline{\omega}(x)+\delta\}| \leq \frac{\varepsilon}{2}.$$

Hence we conclude that $|F_n|_\Omega \leq \frac{\varepsilon}{2}$.

Since $\|v_n-\underline{\omega}\|_{X_0^\alpha}\to 0$ as $n\to\infty$, in particular by the trace embedding, $\|v_n-\underline{\omega}\|_{L^2(\Omega)}\to 0$. We obtain that, for $n\geq n_0$ large,

$$\frac{\delta^2\varepsilon}{2} \geq \int_{\mathcal{C}_\Omega}|v_n-\underline{\omega}|^2dx \geq \int_{E_n}|v_n-\underline{\omega}|^2dx \geq \delta^2|E_n|_\Omega.$$

Therefore $|E_n|_\Omega\leq\dfrac{\varepsilon}{2}$. Since $\widetilde{S}_n\subset F_n\cup E_n$ we conclude that $|\widetilde{S}_n|_\Omega\leq\varepsilon$ for $n\leq n_0$. Hence $|\widetilde{S}_n|_\Omega\to 0$ as $n\to\infty$ and the claim follows. □

4.4. **Proof of Theorems 1.3-1.4 and further results.** We start with the uniform L^∞-estimates for solutions to problem (1.1) in its local version given by (P_λ).

Theorem 4.10. *Assume $\alpha\geq 1$, $p<\frac{N+\alpha}{N-\alpha}$ and $N\geq 2$. Then there exists a constant $C=C(p,\Omega)>0$ such that every solution to problem (P_λ) satisfies*

$$\|w\|_\infty \leq C,$$

for every $0 \le \lambda \le \Lambda$.

The proof is based in a rescaling method and the following two nonexistence results, proved in [32]:

Theorem 4.11. *Let* $\alpha \ge 1$. *Then the problem in the half-space* $\mathbb{R}^{N+1}_+$,

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial w}{\partial \nu^\alpha}(x) &= w^p(x,0) & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N \end{cases}$$

has no solution provided $p < \frac{N+\alpha}{N-\alpha}$, *if* $N \ge 2$, *or for every* p *if* $N = 1$.

Theorem 4.12. *The problem in the first quarter*

$$\mathbb{R}^{N+1}_{++} = \{z = (x', x_N, y) \,|\, x' \in \mathbb{R}^{N-1},\, x_N > 0,\, y > 0\},$$

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0, & x_N > 0,\, y > 0, \\ \dfrac{\partial w}{\partial \nu^\alpha}(x', x_N) &= w^p(x', x_N, 0), & \\ w(x', 0, y) &= 0, & \end{cases}$$

has no positive bounded solution provided $p < \frac{N+\alpha}{N-\alpha}$.

Proof of Theorem 4.10. Assume by contradiction that there exists a sequence $\{w_n\} \subset X_0^\alpha(\mathcal{C}_\Omega)$ of solutions to (P_λ) verifying that $M_n = \|w_n\|_\infty \to \infty$, as $n \to \infty$. By the Maximum Principle, the maximum of w_n is attained at a point $(x_n, 0)$ where $x_n \in \Omega$. We define $\Omega_n = \frac{1}{\mu_n}(\Omega - x_n)$, with $\mu_n = M_n^{(1-p)/\alpha}$, i.e., we center at x_n and dilate by $\frac{1}{\mu_n} \to \infty$ as $n \to \infty$.

We consider the scaled functions

$$v_n(x,y) = \frac{w_n(x_n + \mu_n x, \mu_n y)}{M_n}, \quad \text{for } x \in \Omega_n,\ y \ge 0.$$

It is clear that $\|v_n\| \le 1$, $v_n(0,0) = 1$ and moreover

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla v_n) &= 0 & \text{in } \mathcal{C}_{\Omega_n}, \\ v_n &= 0 & \text{on } \partial_L\mathcal{C}_{\Omega_n}, \\ \dfrac{\partial v_n}{\partial \nu^\alpha} &= \lambda M_n^{q-p} v_n^q + v_n^p & \text{in } \Omega_n \times \{0\}. \end{cases} \tag{4.8}$$

By the Arzelà-Ascoli Theorem, there exists a subsequence, denoted again by v_n, which converges to some function v as $n \to \infty$. In order to see the problem satisfied by v we pass to the limit in the weak formulation of (4.8). We observe that $\|v_n\|_\infty \le 1$ implies $\|v_n\|_{X_0^\alpha(\mathcal{C}_\Omega)} \le C$.

We define $d_n = \text{dist }(x_n, \partial\Omega)$, then there are two possibilities as $n \to \infty$ according the behaviour of the ratio $\frac{d_n}{\mu_n}$:

(1) $\left\{\dfrac{d_n}{\mu_n}\right\}_n$ is not bounded.

(2) $\left\{\dfrac{d_n}{\mu_n}\right\}_n$ remains bounded.

In the first case, since $B_{d_n/\mu_n}(0) \subset \Omega_n$, for another subsequence if necessary, it is clear that Ω_n tends to $\mathbb{R}^N$ and the limit function v is a solution to

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla v) &= 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \dfrac{\partial v}{\partial \nu^\alpha} &= v^p & \text{on } \partial\mathbb{R}^{N+1}_+. \end{cases}$$

Moreover, $v(0,0) = 1$ and $v > 0$ which is a contradiction with Theorem 4.11.

In the second case, we may assume, again for another subsequence if necessary, that $\frac{d_n}{\mu_n} \to s \geq 0$ as $n \to \infty$. As a consequence, passing to the limit, the domains Ω_n converge (up to a rotation) to some half-space $H_s = \{x \in \mathbb{R}^N : x_N > -s\}$. We obtain here that v is a solution to

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla v) &= 0 & \text{in } H_s \times (0,\infty), \\ \dfrac{\partial v}{\partial \nu^\alpha} &= v^p & \text{on } H_s \times \{0\}, \end{cases}$$

with $\|v\|_\infty = 1$, $v(0,0) = 1$. In the case $s = 0$ this is a contradiction with the continuity of v. If $s > 0$, the contradiction comes from Theorem 4.12. $\square$

We next prove Theorem 1.3 in its local version.

Theorem 4.13. *There exists at most one solution to problem (P_λ) with small norm.*

We follow closely the arguments in [4], so we establish the following previous result:

Lemma 4.14. *Let z be the unique solution to problem (4.6). There exists a constant $\beta > 0$ such that*

$$\|\phi\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - q\int_\Omega z^{q-1}\phi^2\,dx \geq \beta\|\phi\|^2_{L^2(\Omega)}, \quad \forall \phi \in X_0^\alpha(\mathcal{C}_\Omega). \tag{4.9}$$

Proof. We recall that z can be obtained by minimization

$$\min\left\{\frac{1}{2}\|\omega\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \frac{1}{q+1}\|w\|^{q+1}_{L^{q+1}(\Omega)} : \quad \omega \in X_0^\alpha(\mathcal{C}_\Omega)\right\}.$$

As a consequence,

$$\|\phi\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - q\int_\Omega z^{q-1}\phi^2\,dx \geq 0, \quad \forall \phi \in X_0^\alpha(\mathcal{C}_\Omega).$$

This implies that the first eigenvalue ρ_1 of the linearized problem

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla \phi) &= 0, & \text{in } \mathcal{C}_\Omega, \\ \phi &= 0, & \text{on } \partial_L\mathcal{C}_\Omega, \\ \dfrac{\partial \phi}{\partial \nu^\alpha} - qz^{q-1}\phi &= \rho\phi, & \text{on } \Omega \times \{0\}, \end{cases}$$

is nonnegative.

Assume first that $\rho_1 = 0$ and let φ be a corresponding eigenfunction. Taking into account that z is the solution to (4.6) we obtain that

$$q\int_\Omega z^q\varphi\,dx = \int_\Omega z^q\varphi\,dx$$

which is a contradiction.

Hence $\rho_1 > 0$, which proves (4.9). □

Proof of Theorem 4.13. Consider $A > 0$ such that $pA^{p-1} < \beta$, where β is given in (4.9). Now we prove that problem (P_λ) has at most one solution with L^∞-norm less than A.

Assume by contradiction that (P_λ) has a second solution $w = w_\lambda + v$ verifying $\|w\|_\infty < A$. Since w_λ is the minimal solution, it follows that $v > 0$ in $\Omega \times [0, \infty)$. We define now $\eta = \lambda^{\frac{1}{1-q}} z$, where z is the solution to (4.6). Then it verifies $-\mathrm{div}(y^{1-\alpha}\nabla\eta) = 0$, with boundary condition $\lambda\eta^q$. Moreover, w_λ is a supersolution to the problem that η verifies. Then by comparison, Lemma 4.3, applied with $f(t) = \lambda t^q$, $v = \eta$ and $w = w_\lambda$, we get

$$w_\lambda \geq \lambda^{\frac{1}{1-q}} z \qquad \text{on } \Omega \times \{0\}. \tag{4.10}$$

Since $w = w_\lambda + v$ is solution to (P_λ) we have, on $\Omega \times \{0\}$,

$$\frac{\partial(w_\lambda + v)}{\partial\nu^\alpha} = \lambda(w_\lambda + v)^q + (w_\lambda + v)^p \leq \lambda w_\lambda^q + \lambda q w_\lambda^{q-1} v + (w_\lambda + v)^p,$$

where the inequality is a consequence of the concavity, hence

$$\frac{\partial v}{\partial\nu^\alpha} \leq \lambda q w_\lambda^{q-1} v + (w_\lambda + v)^p - w_\lambda^p.$$

Moreover, (4.10) implies $w_\lambda^{q-1} \leq \lambda^{-1} z^{q-1}$. From the previous two inequalities we get

$$\frac{\partial v}{\partial\nu^\alpha} \leq q z^{q-1} v + (w_\lambda + v)^p - w_\lambda^p.$$

Using that $\|w_\lambda + v\|_\infty \leq A$, we obtain $(w_\lambda + v)^p - w_\lambda^p \leq pA^{p-1} v$. As a consequence,

$$\frac{\partial v}{\partial\nu^\alpha} - q z^{q-1} v \leq pA^{p-1} v.$$

Taking v as a test function and $\phi = v$ in (4.9) we arrive to

$$\beta \int_\Omega v^2\, dx \leq pA^{p-1} \int_\Omega v^2\, dx.$$

Since $pA^{p-1} < \beta$ we conclude that $v \equiv 0$, which gives the desired contradiction. □

Remark 4.3. This proof also provides the asymptotic behavior of w_λ near $\lambda = 0$, namely $w_\lambda \approx \lambda^{\frac{1}{1-q}} z$, where z is the unique solution to problem (4.6).

Acknowledgments.
C.B. and A.dP. are partially supported by Spanish Project MTM2008-06326-C02-02; E.C. is partially supported by Spanish Project MTM2009-10878.

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A concave-convex elliptic problem involving the fractional Laplacian

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October 22, 2010

Abstract

We study a nonlinear elliptic problem defined in a bounded domain involving fractional powers of the Laplacian operator together with a concave-convex term. We characterize completely the range of parameters for which solutions of the problem exist and prove a multiplicity result.

Keywords: semilinear elliptic equations, fractional Laplacian, Sobolev trace inequality.
2000 Mathematics Subject classification: 35R11, 35J20, 46E35,

1 Introduction

In the past decades the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been widely investigated. See [3] for a survey, and for example the list (far from complete) [4, 12, 33] for more specific problems, where different nonlinearities and different classes of domains, bounded or not, are considered. Other different diffusion operators, like the p–Laplacian, fully nonlinear operators, etc, have been also treated, see for example [8, 15, 25] and the references there in. We deal here with a nonlocal version of the above problem, for a particular type of nonlinearities, i.e., we study a concave-convex problem involving the fractional Laplacian operator

$$\begin{cases} (-\Delta)^{\alpha/2} u = \lambda u^q + u^p, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $0 < \alpha < 2$, $0 < q < 1 < p < \frac{N+\alpha}{N-\alpha}$, $N > \alpha$, $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$ a smooth bounded domain.

The nonlocal operator $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^N$ is defined on the Schwartz class of functions $g \in \mathcal{S}$ through the Fourier transform,

$$[(-\Delta)^{\alpha/2} g]^\wedge(\xi) = (2\pi|\xi|)^\alpha \, \widehat{g}(\xi),$$

or via the Riesz potential, see for example [30, 37]. Observe that $\alpha = 2$ corresponds to the standard local Laplacian.

This type of diffusion operators arises in several areas such as physics, probability and finance, see for instance [6, 7, 21, 41]. In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy process, [7].

There is another way of defining this operator. In fact, in the case $\alpha = 1$ there is an explicit form of calculating the half-Laplacian acting on a function u in the whole space $\mathbb{R}^N$, as the normal derivative on the boundary of its harmonic extension to the upper half-space $\mathbb{R}^{N+1}_+$, the so-called Dirichlet to Neumann operator. The "α derivative" $(-\Delta)^{\alpha/2}$ can be characterized in a similar way, defining the α-harmonic extension to the upper half-space, see [17] and Section A for details. This extension is commonly used in the recent literature since it allows to write nonlocal problems in a local way and this permits to use the variational techniques for these kind of problems.

In the case of the operator defined in bounded domains Ω, the above characterization has to be adapted. The fractional powers of a linear positive operator in Ω are defined by means of the spectral decomposition. In [14], the authors consider the fractional operator $(-\Delta)^{1/2}$ defined using the mentioned Dirichlet to Neumann operator, but restricted to the cylinder $\Omega \times \mathbb{R}_+ \subset \mathbb{R}^{N+1}_+$, and show that this definition is coherent with the spectral one, see also [38] for the case $\alpha \neq 1$. We recall that this is not the unique possibility of defining a nonlocal operator related to the fractional Laplacian in a bounded domain. See for instance the definition of the so called *regional fractional Laplacian* in [9, 29], where the authors consider the Riesz integral restricted to the domain Ω. This leads to a different operator related to a Neumann problem.

As to the concave-convex nonlinearity, there is a huge amount of results involving different (local) operators, see for instance [1, 4, 8, 18, 20, 25]. We quoted the work [4] from where some ideas are used in the present paper. In most of the problems considered in those papers a critical exponent appears, which generically separates the range where compactness results can be applied or can not (in the fully nonlinear case the situation is slightly different, but still a critical exponent appears, [18]). In our case, the critical exponent with respect to the corresponding Sobolev embedding is given by $2^*_\alpha = \frac{2N}{N-\alpha}$. This is a reason why problem (1.1) is studied in the subcritical

case $p < 2^*_\alpha - 1 = \frac{N+\alpha}{N-\alpha}$; see also the nonexistence result for supercritical nonlinearities in Corollary 3.5.

The main results we prove characterize the existence of solutions of (1.1) in terms of the parameter λ. A competition between the sublinear and superlinear terms plays a role, which leads to different results concerning existence and multiplicity of solutions, among others. By a solution we mean an energy solution, see the precise definition in Section 3.

Theorem 1.1 *There exists $\Lambda > 0$ such that for Problem* (1.1) *there holds:*

1. *If $0 < \lambda < \Lambda$ there is a minimal solution. Moreover, the family of minimal solutions is increasing with respect to λ.*
2. *If $\lambda = \Lambda$ there is at least one solution.*
3. *If $\lambda > \Lambda$ there is no solution.*
4. *For any $0 < \lambda < \Lambda$ there exist at least two solutions.*

For $\alpha \in [1, 2)$ and p subcritical, we also prove that there exists a universal L^∞-bound for every solution independently of λ.

Theorem 1.2 *Let $\alpha \geq 1$. Then there exists a constant $C > 0$ such that, for any $0 < \lambda \leq \Lambda$, every solution to Problem* (1.1) *satisfies*

$$\|u\|_\infty \leq C.$$

The proof of this last result relies on the classical argument of rescaling introduced in [27] which yields to problems on unbounded domains, which require some Liouville-type results, which can be seen in [34]. This is the point where the restriction $\alpha \geq 1$ appears.

The paper is organized as follows: we devote Section 2 to settle the preliminaries, such as define properly the fractional Laplacian in a bounded domain, by means of the use of the α-harmonic extension, and study an associated linear equation in the local version. The main section, Section 3, contains the results related to the nonlocal nonlinear problem (1.1), where we prove Theorems 1.1–1.2. We include an appendix where some trace inequality, needed in the proofs, is obtained, with sharp constant.

2 The fractional Laplacian in a bounded domain

2.1 The α-harmonic extension

To define the fractional Laplacian in a bounded domain we follow [14], see also [38]. The idea is to use the α-harmonic extension introduced in [17]

to define the same operator in the whole space, see also the Appendix, but restricted to our bounded domain. To this aim we consider the cylinder

$$C_\Omega = \{(x,y) : x \in \Omega,\ y \in \mathbb{R}_+\} \subset \mathbb{R}^{N+1}_+ ,$$

and denote by $\partial_L C_\Omega$ its lateral boundary.

We first define the extension operator and fractional Laplacian for smooth functions.

Definition 2.1 *Given a regular function u, we define its α-harmonic extension $w = \mathrm{E}_\alpha(u)$ to the cylinder C_Ω as the solution to the problem*

$$\left\{ \begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla w) &=& 0 & \text{in } C_\Omega, \\ w &=& 0 & \text{on } \partial_L C_\Omega, \\ w &=& u & \text{on } \Omega. \end{array} \right. \tag{2.1}$$

As in the whole space, there is also a Poisson formula for the extension operator in a bounded domain, defined through the Laplace transform and the heat semigroup generator $e^{t\Delta}$, see [38] for details.

Definition 2.2 *The fractional operator $(-\Delta)^{\alpha/2}$ in Ω, acting on a regular function u, is defined by*

$$(-\Delta)^{\alpha/2} u(x) = -\frac{1}{\kappa_\alpha} \lim_{y\to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}(x,y), \tag{2.2}$$

where $w = \mathrm{E}_\alpha(u)$ and κ_α is given as in (A.1) *in the Appendix.*

This operator can be extended by density to a fractional Sobolev space.

2.2 Spectral decomposition

It is classical that the powers of a positive operator in a bounded domain (or in an unbounded domain provided the spectrum is discrete) are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. We show next that in this case this is coherent with the Dirichlet-Neumann operator defined above. Let (φ_j, λ_j) be the eigenfunctions and eigenvectors of $-\Delta$ in Ω with Dirichlet boundary data. Define the space of functions defined in our domain Ω,

$$H_0^{\alpha/2}(\Omega) = \{u = \sum a_j \varphi_j \in L^2(\Omega) \ : \ \|u\|_{H_0^{\alpha/2}} = \Big(\sum a_j^2 \lambda_j^{\alpha/2}\Big)^{1/2} < \infty\},$$

its topological dual $H^{-\alpha/2}(\Omega)$, and also the energy space of functions defined in the cylinder C_Ω,

$$X_0^\alpha(C_\Omega) = \{z \in L^2(C_\Omega) \,:\, z = 0 \text{ on } \partial_L C_\Omega,\ \|z\|^2_{X_0^\alpha} = \left(\int_{C_\Omega} y^{1-\alpha}|\nabla z|^2\right)^{1/2} < \infty\}.$$

Lemma 2.3 1. *The eigenfunctions and eigenvectors of $(-\Delta)^{\alpha/2}$ in Ω with Dirichlet boundary data are given by $(\varphi_j, \lambda_j^{\alpha/2})$.*

2. *If $u = \sum a_j \varphi_j \in H_0^{\alpha/2}(\Omega)$ then $\mathrm{E}_\alpha(u) \in X_0^\alpha(C_\Omega)$ and*

$$\mathrm{E}_\alpha(u)(x,y) = \sum a_j \varphi_j(x) \psi(\lambda_j^{1/2} y),$$

where $\psi(s)$ solves the problem

$$\begin{cases} \psi'' + \dfrac{(1-\alpha)}{s}\psi' &= \psi, \qquad s > 0, \\ -\lim\limits_{s\to 0^+} s^{1-\alpha}\psi'(s) &= \kappa_\alpha, \\ \psi(0) &= 1. \end{cases}$$

3. *In the same hypotheses, $(-\Delta)^{\alpha/2} u \in H^{-\alpha/2}(\Omega)$, and $(-\Delta)^{\alpha/2} u = \sum a_j \lambda_j^{\alpha/2} \varphi_j$.*

4. *It holds*

$$\|(-\Delta)^{\alpha/2} u\|_{H^{-\alpha/2}} = \|(-\Delta)^{\alpha/4} u\|_{L^2} = \|u\|_{H_0^{\alpha/2}} = \kappa_\alpha^{-1/2} \|\mathrm{E}_\alpha(u)\|_{X_0^\alpha}.$$

The proof of this result is straightforward. The function ψ coincides with the solution ϕ_α in problem (A.8). The calculation of the norms is also straightforward: Using the orthogonality of the family $\{\varphi_j\}$, together with $\int_\Omega \varphi_j^2 = 1$, $\int_\Omega |\nabla \varphi_j|^2 = \lambda_j$, and (A.10), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} y^{1-\alpha} |\nabla \mathrm{E}_\alpha(u)(x,y)|^2 \, dxdy \\
&= \int_0^\infty y^{1-\alpha} \int_\Omega \left(\sum a_j^2 |\nabla \varphi_j(x)|^2 \psi(\lambda_j^{1/2} y)^2 + a_j^2 \lambda_j \varphi_j(x)^2 (\psi'(\lambda_j^{1/2} y))^2\right) dxdy \\
&= \int_0^\infty y^{1-\alpha} \sum a_j^2 \lambda_j \left(\psi(\lambda_j^{1/2} y)^2 + (\psi'(\lambda_j^{1/2} y))^2\right) dy \\
&= \sum a_j^2 \lambda_j^{\alpha/2} \int_0^\infty s^{1-\alpha} \left(\psi(s)^2 + (\psi'(s))^2\right) ds = \kappa_\alpha \sum a_j^2 \lambda_j^{\alpha/2} \\
&= \kappa_\alpha \sum (a_j \lambda_j^{\alpha/4})^2.
\end{aligned}$$

2.3 The linear problem

We now use the extension problem (2.1) and the expression (2.2) to reformulate the nonlocal problems in a local way. Let g be a regular function and consider the following problems, the nonlocal problem

$$\left\{ \begin{array}{rcll} (-\Delta)^{\alpha/2} u & = & g(x) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{2.3}$$

and the corresponding local one

$$\left\{ \begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla w) & = & 0 & \text{in } C_\Omega, \\ w & = & 0 & \text{on } \partial_L C_\Omega, \\ -\dfrac{1}{\kappa_\alpha} \lim\limits_{y\to 0^+} y^{1-\alpha} \dfrac{\partial w}{\partial y} & = & g(x) & \text{on } \Omega. \end{array} \right. \tag{2.4}$$

We want to define the concept of solution to (2.3), which is done in terms of the solution to problem (2.4).

Definition 2.4 *We say that $w \in X_0^\alpha(C_\Omega)$ is an energy solution to problem* (2.4)*, if for every function $\varphi \in \mathcal{C}_0^1(C_\Omega)$ it holds*

$$\int_{C_\Omega} y^{1-\alpha} \langle \nabla w(x,y), \nabla\varphi(x,y) \rangle \, dxdy = \int_\Omega \kappa_\alpha g(x)\varphi(x,0)\, dx. \tag{2.5}$$

In fact more general test functions can be used in the above formula, whenever the integrals make sense. A supersolution (subsolution) is a function that verifies (2.5) with equality replaced by $\geq$ ($\leq$) for every nonnegative test function.

Definition 2.5 *We say that $u \in H_0^{\alpha/2}(\Omega)$ is an energy solution to problem* (2.3) *if it is the trace on Ω of a function w which is an energy solution to problem* (2.4)*.*

It is clear that a solution exists for instance for every $g \in H^{-\alpha/2}(\Omega)$. In order to deal with problem (2.4) we will assume, without loss of generality, $\kappa_\alpha = 1$, by changing the function g.

In [13] this linear problem is also mentioned. There some results are obtained using the theory of degenerate elliptic equations developed in [24], in particular a regularity result for bounded solutions to this problem is obtained in [13]. We prove here that the solutions are in fact bounded if g satisfies a minimal integrability condition.

Theorem 2.6 *Let w be a solution to problem* (2.4)*. If $g \in L^r(\Omega)$, $r > \frac{N}{\alpha}$, then $w \in L^\infty(C_\Omega)$.*

Proof. The proof follows from the well-known Moser's iterative technique, that we take from [28, Theorem 8.15], and uses the trace inequality (A.12). Without loss of generality we may assume $w \geq 0$, and this simplifies notation. The general case is obtained in a similar way.

We define for $\beta \geq 1$ and $K \geq k$ (k to be chosen later) a $\mathcal{C}^1([k,\infty))$ function H, as follows:

$$H(z) = \begin{cases} z^\beta - k^\beta, & z \in [k,K], \\ \text{linear}, & z > K. \end{cases}$$

Let us also define $v = w + k$, $\nu = \mathrm{Tr}(v)$, and choose as test function φ,

$$\varphi = G(v) = \int_k^v |H'(s)|^2\, ds, \qquad \nabla\varphi = |H'(v)|^2 \nabla v.$$

Observe that it is an admissible test function, though it is not $\mathcal{C}^1$. Replacing this test function into the definition of energy solution we obtain on one hand:

$$\begin{aligned} \int_{C_\Omega} y^{1-\alpha}\langle \nabla w, \nabla\varphi\rangle\, dxdy &= \int_{C_\Omega} y^{1-\alpha}|\nabla v|^2 |H'(v)|^2\, dxdy \\ &= \int_{C_\Omega} y^{1-\alpha}|\nabla H(v)|^2\, dxdy \\ &\geq \left(\int_\Omega |H(\nu)|^{\frac{2N}{N-\alpha}}\, dx\right)^{\frac{N-\alpha}{N}} = \|H(\nu)\|^2_{\frac{2N}{N-\alpha}}, \end{aligned} \tag{2.6}$$

where the last inequality follows by (A.12).

On the other hand

$$\begin{aligned} \int_\Omega g(x)\varphi(x,0)\, dx &= \int_\Omega g(x) G(\nu)\, dx \leq \int_\Omega g(x)\nu G'(\nu)\, dx \\ &\leq \frac{1}{k}\int_\Omega g(x)\nu^2 |H'(\nu)|^2\, dx = \frac{1}{k}\int_\Omega g(x)|\nu H'(\nu)|^2\, dx. \end{aligned} \tag{2.7}$$

Inequality (2.6) together with (2.7), leads to

$$\|H(\nu)\|_{\frac{2N}{N-\alpha}} \leq \left(\frac{1}{k}\|g\|_r\right)^{1/2} \|(\nu H'(\nu))^2\|^{1/2}_{\frac{r}{r-1}} = \|\nu H'(\nu)\|_{\frac{2r}{r-1}}, \tag{2.8}$$

by choosing $k = \|g\|_r$. Letting $K \to \infty$ in the definition of H, the inequality (2.8) becomes

$$\|\nu\|_{\frac{2N\beta}{N-\alpha}} \leq \|\nu\|_{\frac{2r\beta}{r-1}}.$$

Hence for all $\beta \geq 1$ the inclusion $\nu \in L^{\frac{2r\beta}{r-1}}(\Omega)$ implies the stronger inclusion $\nu \in L^{\frac{2N\beta}{N-\alpha}}(\Omega)$, since $\frac{2N\beta}{N-\alpha} > \frac{2r\beta}{r-1}$ provided $r > \frac{N}{\alpha}$. The result follows now,

as in [28], by an iteration argument, starting with $\beta = \frac{N(r-1)}{r(N-\alpha)} > 1$ and $\nu \in L^{\frac{2N}{N-\alpha}}(\Omega)$. This gives $\nu \in L^\infty(\Omega)$, and then $w \in L^\infty(C_\Omega)$. In fact we get the estimate

$$\|w\|_\infty \leq c(\|w\|_{X^\alpha} + \|g\|_r).$$

$\square$

Corollary 2.7 *Let w be a solution to problem (2.4). If $g \in L^\infty(\Omega)$, then $w \in \mathcal{C}^\gamma(\overline{C_\Omega})$ for some $\gamma \in (0,1)$.*

Proof. Using Theorem 2.6, the result follows directly from [13, Lemma 4.4], where it is proved that any bounded solution to problem (2.4) with a bounded g is $\mathcal{C}^\gamma$. $\square$

3 The nonlinear nonlocal problem

3.1 The local realization

We deal now with the core of the paper; i.e. the study of the nonlocal problem (1.1). We write that problem in local version in the following way: a solution to problem (1.1) is a function $u = \mathrm{Tr}(w)$, the trace of w on $\Omega \times \{y = 0\}$, where w solves the local problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla w) & = & 0 & \text{in } C_\Omega, \\ w & = & 0 & \text{on } \partial_L C_\Omega, \\ \dfrac{\partial w}{\partial \nu^\alpha} & = & f(w) & \text{in } \Omega, \end{array} \right. \tag{3.1}$$

where

$$\frac{\partial w}{\partial \nu^\alpha}(x) = -\lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y). \tag{3.2}$$

Note. In order to simplify notation in what follows we will denote, when no confusio arises, w for the function defined in the cylinder C_Ω as well as for its trace $\mathrm{Tr}(w)$ on $\Omega \times \{y = 0\}$.

As we have said, we will focus on the particular nonlinearity

$$f(s) = f_\lambda(s) = \lambda s^q + s^p. \tag{3.3}$$

However many auxiliary results will be proved for more general reactions f satisfying the growth condition

$$0 \leq f(s) \leq c(1 + |s|^p), \qquad \text{for some } p > 0. \tag{3.4}$$

Remark 3.1 In the definition (3.2) we have neglected the constant κ_α appearing in (2.2) by a simple rescaling. Therefore, the results on the coefficient λ for the local problem (3.1)–(3.3) in this section are translated into problem (1.1) with λ multiplied by $\kappa_\alpha^{p(q-1)-1}$.

Following Definition 2.4, we say that $w \in X_0^\alpha(\mathcal{C}_\Omega)$ is an energy solution of (3.1) if the following identity holds

$$\int_{C_\Omega} y^{1-\alpha}\langle \nabla w, \nabla\varphi\rangle \, dxdy = \int_\Omega f(w)\varphi \, dx$$

for every regular test function φ. In the analogous way we define sub- and supersolution.

We consider now the functional

$$J(w) = \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 \, dxdy - \int_\Omega F(w)\, dx,$$

where $F(s) = \int_0^s f(\tau)\, d\tau$. For simplicity of notation, we define $f(s) = 0$ for $s \le 0$. Recall that the trace satisfies $w \in L^r(\Omega)$, (again this means $\mathrm{Tr}(w) \in L^r(\Omega)$), for every $1 \le r \le \frac{2N}{N-\alpha}$ if $N > \alpha$, $1 < r \le \infty$ if $N \le \alpha$. In particular if $p \le \frac{N+\alpha}{N-\alpha}$, and f verifies (3.4) then $F(w) \in L^1(\Omega)$, and the functional is well defined and bounded from below.

It is well known that critical points of J are solutions to (3.1) with a general reaction f. We consider also the minimization problem

$$I = \inf\Big\{ \int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 \, dxdy \,:\, w \in X_0^\alpha(C_\Omega),\ \int_\Omega F(w)\, dx = 1\Big\},$$

for which, by classical variational techniques, one has that below the critical exponent the infimum I is achieved. This gives a nonnegative solution. Later on we will see that this infimum is positive provided $\lambda > 0$ is small enough. On the contrary, for λ large enough the infimum is the trivial solution.

We now establish two preliminary results. The first one is a classical procedure of sub- and supersolutions to obtain a solution. We omit its proof.

Lemma 3.1 *Assume there exist a subsolution w_1 and a supersolution w_2 to problem* (3.1) *verifying $w_1 \le w_2$. Then there also exists a solution w satisfying $w_1 \le w \le w_2$ in C_Ω.*

The second one is a comparison result for concave nonlinearities. The proof follows the lines of the corresponding one for the Laplacian performed in [10].

Lemma 3.2 *Assume the function $f(t)/t$ is decreasing for $t > 0$ and consider $w_1, w_2 \in X_0^\alpha(C_\Omega)$ positive subsolution and supersolution, respectively, to problem* (3.1)*. Then $w_1 \le w_2$ in $\overline{C_\Omega}$.*

Proof. By definition we have, for the nonnegative test functions φ_1 and φ_2 to be chosen in an appropriate way,

$$\int_{C_\Omega} y^{1-\alpha} \big\langle \nabla w_1, \nabla \varphi_1 \big\rangle \, dxdy \le \int_\Omega f(w_1)\varphi_1 \, dx,$$
$$\int_{C_\Omega} y^{1-\alpha} \big\langle \nabla w_2, \nabla \varphi_2 \big\rangle \, dxdy \ge \int_\Omega f(w_2)\varphi_2 \, dx.$$

Now let $\theta(t)$ be a smooth nondecreasing function such that $\theta(t) = 0$ for $t \le 0$, $\theta(t) = 1$ for $t \ge 1$, and set $\theta_\varepsilon(t) = \theta(\frac{t}{\varepsilon})$. If we put, in the above inequalities

$$\varphi_1 = w_2\, \theta_\varepsilon(w_1 - w_2), \qquad \varphi_2 = w_1\, \theta_\varepsilon(w_1 - w_2),$$

we get

$$I_1 \ge \int_\Omega w_1 w_2 \Big(\frac{f(w_2)}{w_2} - \frac{f(w_1)}{w_1} \Big) \theta_\varepsilon(w_1 - w_2) \, dx,$$

where

$$I_1 := \int_{C_\Omega} y^{1-\alpha} \big\langle w_1 \nabla w_2 - w_2 \nabla w_1, \nabla(w_1 - w_2) \big\rangle \, \theta_\varepsilon'(w_1 - w_2) \, dxdy.$$

Now we estimate I_1 as follows:

$$\begin{aligned} I_1 \ &\le \int_{C_\Omega} y^{1-\alpha} \big\langle \nabla w_1, (w_1 - w_2)\nabla(w_1 - w_2) \big\rangle \, \theta_\varepsilon'(w_1 - w_2) \, dxdy \\ &= \int_{C_\Omega} y^{1-\alpha} \big\langle \nabla w_1, \nabla \gamma_\varepsilon(w_1 - w_2) \big\rangle \, dxdy \end{aligned}$$

where $\gamma_\epsilon'(t) = t\theta_\varepsilon'(t)$. Therefore, since $0 \le \gamma_\varepsilon \le \varepsilon$, we have

$$I_1 \le \int_\Omega f(w_1)\gamma_\varepsilon(w_1 - w_2) \, dx \le c\varepsilon.$$

We end as in [4]. Letting ε tend to zero, we obtain

$$\int_{\Omega \cap \{w_1 > w_2\}} w_1 w_2 \left(\frac{f(w_2)}{w_2} - \frac{f(w_1)}{w_1} \right) dx \le 0,$$

which together with the hypothesis on f gives $w_1 \le w_2$ in Ω. Comparison in C_Ω follows easily by the maximum principle. $\square$

Now we show that the solutions to problem (3.1)–(3.4) are bounded and Hölder continuous. Later on, in Section 3.4, we will obtain a uniform L^∞-estimate in the case where f is given by (3.3) and the convex power is subcritical.

Proposition 3.3 *Let f satisfy* (3.4) *with $p < \frac{N+\alpha}{N-\alpha}$, and let $w \in X_0^\alpha(C_\Omega)$ be an energy solution to problem* (3.1). *Then $w \in L^\infty(C_\Omega) \cap \mathcal{C}^\gamma(\overline{C_\Omega})$ for some $0 < \gamma < 1$.*

Proof. The proof follows closely the technique of [11]. As in the proof of Theorem 2.6, we assume $w \geq 0$. We consider, formally, the test function $\varphi = w^{\beta - p}$, for some $\beta > p+1$. The justification of the following calculations can be made substituting φ by some approximated truncature. We therefore proceed with the formal analysis. We get, using the trace immersion, the inequality

$$\left(\int_\Omega w^{\frac{(\beta-p+1)N}{N-\alpha}} \right)^{\frac{N-\alpha}{N}} \leq C(\beta, \alpha, N, \Omega) \int_\Omega w^\beta.$$

This estimate allows us to obtain the following iterative process

$$\|w\|_{\beta_{j+1}} \leq C \|w\|_{\beta_j}^{\frac{\beta_j}{\beta_j - p+1}},$$

with $\beta_{j+1} = \frac{N}{N-\alpha}(\beta_j + 1 - p)$. To have $\beta_{j+1} > \beta_j$ we need $\beta_j > \frac{(p-1)N}{\alpha}$. Since $w \in L^{2^*_\alpha}(\Omega)$, starting with $\beta_0 = \frac{2N}{N-\alpha}$, we get the above restriction provided $p < \frac{N+\alpha}{N-\alpha}$. It is clear that in a finite number of steps we get, for $g(x) = f(w(x,0))$, the regularity $g \in L^r$ for some $r > \frac{N}{\alpha}$. As a consequence, we obtain the conclusion applying Theorem 2.6 and Corollary 2.7. □

3.2 A nonexistence result

The following result relies on a the use of a classical Pohozaev type multiplier.

Proposition 3.4 *Assume f is a $\mathcal{C}^1$ function with primitive F, and w is an energy solution to problem* (3.1). *Then the following Pohozaev-type identity holds*

$$\frac{1}{2} \int_{\partial_L C_\Omega} y^{1-\alpha} \big\langle (x,y), \nu \big\rangle |\nabla w|^2 \, d\sigma - N \int_\Omega F(w) \, dx + \frac{N-\alpha}{2} \int_\Omega w f(w) \, dx = 0.$$

Proof. Just use the identity

$$\begin{aligned} &\big\langle (x,y), \nu \big\rangle y^{\alpha-1} \operatorname{div}(y^{1-\alpha} \nabla w) + \operatorname{div} \Big[y^{1-\alpha} \Big(\big\langle (x,y), \nabla w \big\rangle - \frac{1}{2}(x,y) |\nabla w|^2 \Big) \Big] \\ &+ \Big(\frac{N+2-\alpha}{2} - 1 \Big) |\nabla w|^2 = 0, \end{aligned}$$

where ν is the (exterior) normal vector to $\partial\Omega$. It is calculus matter to check this equality. □

As a consequence we obtain a nonexistence result in the supercritical case for domains with particular geometry.

Theorem 3.5 *If Ω is starshaped and the nonlinearity f satisfies the inequality $((N-\alpha)sf(s) - 2NF(s)) \geq 0$, then problem* (3.1) *has no solution. In particular, in the case $f(s) = s^p$ this means that there is no solution for any $p \geq \frac{N+\alpha}{N-\alpha}$.*

The case $\alpha = 1$ has been proved in [14]. The corresponding result for the Laplacian (problem (1.1) with $\alpha = 2$) comes from [36].

3.3 Proof of Theorem 1.1

We prove here Theorem 1.1 in terms of the solution of the local version (3.1). For the sake of readability we split the proof of into several lemmas. From now on we will denote

$$(P_\lambda) \equiv \begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) = 0, & \text{in } C_\Omega, \\ w = 0, & \text{on } \partial_L C_\Omega, \\ \dfrac{\partial w}{\partial \nu^\alpha} = \lambda w^q + w^p, \quad u > 0 & \text{in } \Omega, \end{cases}$$

and consider the associated energy functional

$$J_\lambda(w) = \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy - \int_\Omega F_\lambda(w)\, dx,$$

where

$$F_\lambda(s) = \frac{\lambda}{q+1}s^{q+1} + \frac{1}{p+1}s^{p+1}.$$

Lemma 3.6 *Let Λ be defined by*

$$\Lambda = \sup\{\lambda > 0 \ : \ \textit{Problem } (P_\lambda) \textit{ has solution}\}.$$

Then $0 < \Lambda < \infty$.

Proof. Consider the eigenvalue problem associated to the first eigenvalue λ_1, and let $\varphi_1 > 0$ be an associated eigenfunction, see Lemma 2.3. Then using φ_1 as a test function in (P_λ) we have that

$$\int_\Omega (\lambda w^q + w^p)\varphi_1\, dx = \lambda_1 \int_\Omega w\varphi_1\, dx. \tag{3.5}$$

Since there exist positive constants c, δ such that $\lambda t^q + t^p > c\lambda^\delta t$, for any $t > 0$ we obtain from (3.5) (recall that $u > 0$) that $c\lambda^\delta < \lambda_1$ which implies $\Lambda < \infty$.

To prove $\Lambda > 0$ we use the sub- and supersolution technique to construct a solution for any small λ. In fact a subsolution is obtained as $\underline{w} = \varepsilon\varphi_1$, $\varepsilon > 0$ small. A supersolution is a suitable multiple of the function e solution to

$$\left\{\begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla e) & = & 0 & \text{in } C_\Omega, \\ e & = & 0 & \text{on } \partial_L C_\Omega, \\ \dfrac{\partial e}{\partial \nu^\alpha} & = & 1 & \text{in } \Omega. \end{array}\right.$$

□

This proves the third statement in Theorem 1.1.

Lemma 3.7 *Problem* (P_λ) *has a positive solution for every* $0 < \lambda < \Lambda$. *Moreover, the family* $\{w_\lambda\}$ *of minimal solutions is increasing with respect to* λ.

Remark 3.2 Although this Λ is not exactly the same as that of Theorem 1.1, see Remark 3.1, we have not changed the notation for the sake of simplicity.

Proof. We already proved in the previous lemma that Problem (P_λ) has a solution for every $\lambda > 0$ small. Another way of proving this result is to look at the associated functional J_λ. Using Theorem A.5, we have that this functional verifies

$$\begin{aligned} J_\lambda(w) &= \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy - \int_\Omega F_\lambda(w)\,dx \\ &\geq \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy - \lambda C_1\Big(\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy\Big)^{\frac{q+1}{2}} \\ &\quad - C_2\Big(\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy\Big)^{\frac{p+1}{2}}, \end{aligned}$$

for some positive constants C_1 and C_2. Then for λ small enough there exist two solutions of problem (P_λ), one given by minimization and another one given by the Mountain-Pass Theorem, [5]. The proof is standard, based on the geometry of the function $g(t) = \frac{1}{2}t^2 - \lambda C_1 t^{q+1} - C_2 t^{p+1}$, see for instance [25] for more details.

We now show that there exists a solution for every $\lambda \in (0, \Lambda)$. By definition of Λ, we know that there exists a solution corresponding to any value of λ close to Λ. Let us denote it by μ, and let w_μ be the associated solution. Now w_μ is a supersolution for all problems (P_λ) with $\lambda < \mu$. Take v_λ the unique solution to problem (3.1) with $f(s) = \lambda s^q$. Obviously v_λ is a subsolution to problem (P_λ). By Lemma 3.2 $v_\lambda \leq w_\mu$. Therefore by

Lemma 3.1 we conclude that there is a solution for all $\lambda \in (0, \mu)$, and as a consequence, for the whole open interval $(0, \Lambda)$. Moreover, this solution is the minimal one. The monotonicity follows directly from the comparison lemma. $\square$

This proves the first statement in Theorem 1.1.

Lemma 3.8 *Problem (P_λ) has at least one solution if $\lambda = \Lambda$.*

Proof. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \nearrow \Lambda$. We denote by $w_n = w_{\lambda_n}$ the minimal solution to problem (P_{λ_n}). As in [4], we can prove that the linearized equation at the minimal solution has nonnegative eigenvalues. Then it follows, as in [4] again, $J_{\lambda_n}(w_n) < 0$. Since $J'(w_\lambda) = 0$, one easily gets the bound $\|w_n\|_{X_0^\alpha(C_\Omega)} \leq k$. Hence, there exists a weakly convergent subsequence in $X_0^\alpha(C_\Omega)$ and as a consequence a weak solution of (P_λ) for $\lambda = \Lambda$. $\square$

This proves the second statement in Theorem 1.1. To conclude the proof of that theorem, we show next the existence of a second solution for every $0 < \lambda < \Lambda$. It is essential to have that the first solution is given as a local minimum of the associated functional, J_λ. To prove this last assertion we follow some ideas developed in [2].

Let $\lambda_0 \in (0, \Lambda)$ be fixed and consider $\lambda_0 < \bar{\lambda}_1 < \Lambda$. Take $\phi_0 = w_{\lambda_0}$, $\phi_1 = w_{\bar{\lambda}_1}$ the two minimal solutions to problem (P_λ) with $\lambda = \lambda_0$ and $\lambda = \bar{\lambda}_1$ respectively, then by comparison, $\phi_0 < \phi_1$. We define

$$M = \{w \in X_0^\alpha(C_\Omega) : \, 0 \leq w \leq \phi_1\}.$$

Notice that M is a convex closed set of $X_0^\alpha(C_\Omega)$. Since J_{λ_0} is bounded from bellow in M and it is semicontinuous on M, we get the existence of $\underline{\omega} \in M$ such that $J_{\lambda_0}(\underline{\omega}) = \inf_{w \in M} J_{\lambda_0}(w)$. Let v_0 be the unique positive solution to problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla v_0) & = & 0, & \text{in } \ C_\Omega, \\ v_0 & = & 0, & \text{on } \ \partial_L C_\Omega, \\ \dfrac{\partial v_0}{\partial \nu^\alpha} & = & v_0^q, & \text{in } \ \Omega. \end{array} \right. \tag{3.6}$$

(The existence and uniqueness of this solution is clear). Since for $0 < \varepsilon << \lambda_0$, and $J_{\lambda_0}(\varepsilon v_0) < 0$, we have $\varepsilon v_0 \in M$, then $\underline{\omega} \neq 0$. Therefore $J_{\lambda_0}(\underline{\omega}) < 0$. By arguments similar to those in [39, Theorem 2.4], we obtain that $\underline{\omega}$ is a solution to problem (P_{λ_0}). There are two possibilities:

- If $\underline{\omega} \not\equiv w_{\lambda_0}$, then the result follows.

- If $\underline{\omega} \equiv w_{\lambda_0}$, we have just to prove that $\underline{\omega}$ is a local minimum of J_{λ_0}. Assuming that this is true, the conclusion in part 4 of Theorem 1.1 follows by using a classical argument: The second solution is given by the Mountain Pass Theorem, see for instance [5].

We prove now that the minimal solution w_{λ_0} is in fact a local minimum of J_{λ_0}. We argue by contradiction.

Suppose that $\underline{\omega}$ is not a local minimum of J_{λ_0} in $X_0^\alpha(C_\Omega)$, then there exists a sequence $\{v_n\} \subset X_0^\alpha(C_\Omega)$ such that $\|v_n - \underline{\omega}\|_{X_0^\alpha} \to 0$ and $J_{\lambda_0}(v_n) < J_{\lambda_0}(\underline{\omega})$.

Let $w_n = (v_n - \phi_1)^+$ and $z_n = \max\{0, \min\{v_n, \phi_1\}\}$. It is clear that $z_n \in M$ and

$$z_n(x,y) = \begin{cases} 0 & \text{if } v_n(x,y) \le 0, \\ v_n(x,y) & \text{if } 0 \le v_n(x,y) \le \phi_1(x,y), \\ \phi_1(x,y) & \text{if } \phi_1(x,y) \le v_n(x,y). \end{cases}$$

We set

$$\begin{aligned} &T_n \equiv \{(x,y) \in C_\Omega : \, z_n(x,y) = v_n(x,y)\}, && S_n \equiv \operatorname{supp}(w_n), \\ &\widetilde{T}_n = \overline{T_n} \cap \Omega, && \widetilde{S}_n = S_n \cap \Omega. \end{aligned}$$

Notice that $\operatorname{supp}(v_n^+) = T_n \cup S_n$. We claim that

$$|\widetilde{S}_n|_\Omega \to 0 \quad \text{as } n \to \infty, \tag{3.7}$$

where $|A|_\Omega \equiv \int_\Omega \chi_A(x)\, dx$.

By the definition of F_λ, we set $F_{\lambda_0}(s) = \dfrac{l_0}{q+1} s_+^{q+1} + \dfrac{1}{p+1} s_+^{p+1}$, for $s \in \mathbb{R}$, and get

$$\begin{aligned} J_{l_0}(v_n) =& \frac{1}{2} \int_{C_\Omega} y^{1-\alpha} |\nabla v_n|^2 \, dxdy - \int_\Omega F_{\lambda_0}(v_n)\, dx \\ =& \frac{1}{2} \int_{T_n} y^{1-\alpha} |\nabla z_n|^2 \, dxdy - \int_{\widetilde{T}_n} F_{\lambda_0}(z_n)\, dx + \frac{1}{2} \int_{S_n} y^{1-\alpha} |\nabla v_n|^2 \, dxdy \\ & - \int_{\widetilde{S}_n} F_{\lambda_0}(v_n)\, dx + \frac{1}{2} \int_{C_\Omega} y^{1-\alpha} |\nabla v_n^-|^2 \, dxdy \\ =& \frac{1}{2} \int_{T_n} y^{1-\alpha} |\nabla z_n|^2 \, dxdy - \int_{\widetilde{T}_n} F_{\lambda_0}(z_n)\, dx \\ & + \frac{1}{2} \int_{S_n} y^{1-\alpha} |\nabla (w_n + \phi_1)|^2 \, dxdy - \int_{\widetilde{S}_n} F_{\lambda_0}(w_n + \phi_1)\, dx \\ & + \frac{1}{2} \int_{C_\Omega} y^{1-\alpha} |\nabla v_n^-|^2 \, dxdy. \end{aligned}$$

Since

$$\int_{C_\Omega} y^{1-\alpha}|\nabla z_n|^2\,dxdy = \int_{T_n} y^{1-\alpha}|\nabla v_n|^2\,dxdy + \int_{S_n} y^{1-\alpha}|\nabla \phi_1|^2\,dxdy$$

and

$$\int_\Omega F_{\lambda_0}(z_n)\,dx = \int_{\widetilde{T}_n} F_{\lambda_0}(v_n)\,dx + \int_{\widetilde{S}_n} F_{\lambda_0}(\phi_1)\,dx,$$

by using the fact that ϕ_1 is a supersolution to (P_λ) with $l = l_0$, we conclude that

$$\begin{aligned} J_{l_0}(v_n) &= J_{l_0}(z_n) + \frac{1}{2}\int_{S_n} y^{1-\alpha}(|\nabla(w_n+\phi_1)|^2 - |\nabla\phi_1|^2)\,dxdy \\ &\quad - \int_{\widetilde{S}_n}(F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1))\,dx + \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla v_n^-|^2\,dxdy \\ &\geq J_{l_0}(z_n) + \frac{1}{2}\|w_n\|^2_{X_0^\alpha} + \frac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\ &\quad - \int_\Omega \{F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1) - (F_{\lambda_0})_u(\phi_1)w_n\}\,dx \\ &\geq J_{l_0}(\underline{\omega}) + \frac{1}{2}\|w_n\|^2_{X_0^\alpha} + \frac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\ &\quad - \int_\Omega \{F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1) - (F_{\lambda_0})_u(\phi_1)w_n\}\,dx. \end{aligned}$$

On one hand, taking into account that $0 < q+1 < 2$, one obtains that

$$0 \leq \frac{1}{q+1}(w_n+\phi_1)^{q+1} - \frac{1}{q+1}\phi_1^{q+1} - \phi_1^q w_n \leq \frac{q}{2}\frac{w_n^2}{\phi_1^{1-q}}.$$

The well known Picone's inequality (see [35]) establish:

$$|\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\cdot\nabla v \geq 0,$$

for differentiable functions $v > 0$, $u \geq 0$. In our case, by an approximation argument we get

$$l_0\int_\Omega \frac{w_n^2}{\phi_1^{1-q}}dx \leq \|w_n\|^2_{X_0^\alpha}.$$

On the other hand, since $p+1 > 2$,

$$\begin{aligned} 0 &\leq \frac{1}{p+1}(w_n+\phi_1)^{p+1} - \frac{1}{p+1}\phi_1^{p+1} - \phi_1^p w_n \leq \frac{r}{2}w_n^2(w_n+\phi_1)^{p-1} \\ &\leq C(p)(\phi_1^{p-1}w_n^2 + w_n^{p+1}). \end{aligned}$$

Hence using that $p+1 < 2_\alpha^*$ and the claim (3.7)

$$\int_\Omega \left\{\frac{1}{p+1}(w_n+\phi_1)^{p+1} - \frac{1}{p+1}\phi_1^{p+1} - \phi_1^p w_n\right\}dx \leq o(1)\|w_n\|^2_{X_0^\alpha}.$$

As a consequence we obtain that

$$\begin{aligned} J_{l_0}(v_n) &\geq J_{l_0}(\underline{\omega}) + \frac{1}{2}\|w_n\|^2_{X_0^\alpha}(1-q-o(1)) + \frac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\ &\equiv J_{l_0}(\underline{\omega}) + \frac{1}{2}\|w_n\|^2_{X_0^\alpha}(1-q-o(1)) + o(1). \end{aligned}$$

Since $q < 1$, there results that $J_{l_0}(\underline{\omega}) > J_{l_0}(v_n) \geq J_{l_0}(\underline{\omega})$ for $n > n_0$, a contradiction with the main hypothesis. Hence $\underline{\omega}$ is a minimum.

To finish the proof we have to prove the claim (3.7). For $\varepsilon > 0$ small, and $\delta > 0$ (δ to be chosen later), we consider

$$\begin{aligned} E_n &= \{x \in \Omega : \, v_n(x) \geq \phi_1(x) \quad \wedge \quad \phi_1(x) > \underline{\omega}(x) + \delta\}, \\ F_n &= \{x \in \Omega : \, v_n(x) \geq \phi_1(x) \quad \wedge \quad \phi_1(x) \leq \underline{\omega}(x) + \delta\}. \end{aligned}$$

Using the fact that

$$\begin{aligned} 0 &= |\{x \in \Omega : \, \phi_1(x) < \underline{\omega}(x)\}| = \left| \bigcap_{j=1}^{\infty} \left\{ x \in \Omega : \, \phi_1(x) \leq \underline{\omega}(x) + \frac{1}{j} \right\} \right| \\ &= \lim_{j \to \infty} \left| \left\{ x \in \Omega : \, \phi_1(x) \leq \underline{\omega}(x) + \frac{1}{j} \right\} \right|, \end{aligned}$$

we get for j_0 large enough, that if $\delta < \frac{1}{j_0}$ then

$$|\{x \in \Omega : \, \phi_1(x) \leq \underline{\omega}(x) + \delta\}| \leq \frac{\varepsilon}{2}.$$

Hence we conclude that $|F_n|_\Omega \leq \frac{\varepsilon}{2}$.

Since $\|v_n - \underline{\omega}\|_{X_0^\alpha} \to 0$ as $n \to \infty$, in particular by the trace embedding, $\|v_n - \underline{\omega}\|_{L^2(\Omega)} \to 0$. We obtain that, for $n \geq n_0$ large,

$$\frac{\delta^2 \varepsilon}{2} \geq \int_{C\Omega} |v_n - \underline{\omega}|^2 dx \geq \int_{E_n} |v_n - \underline{\omega}|^2 dx \geq \delta^2 |E_n|_\Omega.$$

Therefore $|E_n|_\Omega \leq \dfrac{\varepsilon}{2}$. Since $\widetilde{S}_n \subset F_n \cup E_n$ we conclude that $|\widetilde{S}_n|_\Omega \leq \varepsilon$ for $n \leq n_0$. Hence $|\widetilde{S}_n|_\Omega \to 0$ as $n \to \infty$ and the claim follows. $\square$

3.4 Proof of Theorem 1.2 and further results

We start with the uniform L^∞-estimates for solutions to problem (1.1) in its local version given by (P_λ).

Theorem 3.9 *Assume $\alpha \geq 1$, $p < \frac{N+\alpha}{N-\alpha}$ and $N \geq 2$. Then there exists a constant $C = C(p, \Omega) > 0$ such that every solution to problem (P_λ) satisfies*

$$\|w\|_\infty \leq C,$$

for every $0 \leq \lambda \leq \Lambda$.

The proof is based in a rescaling method and the following two nonexistence results, proved in [34]:

Theorem 3.10 *Let $\alpha \geq 1$. Then the problem in the half-space $\mathbb{R}^{N+1}_+$,*

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \dfrac{\partial w}{\partial \nu^\alpha}(x) &= w^p(x,0) & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N \end{cases}$$

has no solution provided $p < \frac{N+\alpha}{N-\alpha}$, if $N \geq 2$, or for every p if $N = 1$.

Theorem 3.11 *The problem in the first quarter*

$$\mathbb{R}^{N+1}_{++} = \{z = (x', x_N, y) \mid x' \in \mathbb{R}^{N-1},\ x_N > 0,\ y > 0\},$$

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0, & x_N > 0,\ y > 0, \\ \dfrac{\partial w}{\partial \nu^\alpha}(x', x_N) &= w^p(x', x_N, 0), & \\ w(x', 0, y) &= 0, & \end{cases}$$

has no positive bounded solution provided $p < \frac{N+\alpha}{N-\alpha}$.

Proof of Theorem 3.9. Assume by contradiction that there exists a sequence $\{w_n\} \subset X_0^\alpha(C_\Omega)$ of solutions to (P_λ) verifying that $M_n = \|w_n\|_\infty \to \infty$, as $n \to \infty$. By the Maximum Principle, which holds for our problem, see [24], the maximum of w_n is attained at a point $(x_n, 0)$ where $x_n \in \Omega$. We define $\Omega_n = \frac{1}{\mu_n}(\Omega - x_n)$, with $\mu_n = M_n^{(1-p)/\alpha}$, i.e., we center at x_n and dilate by $\frac{1}{\mu_n} \to \infty$ as $n \to \infty$.

We consider the scaled functions

$$v_n(x,y) = \frac{w_n(x_n + \mu_n x, \mu_n y)}{M_n}, \quad \text{for } x \in \Omega_n,\ y \geq 0.$$

It is clear that $\|v_n\| \leq 1$, $v_n(0,0) = 1$ and moreover

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla v_n) &= 0 & \text{in } C_{\Omega_n}, \\ v_n &= 0 & \text{on } \partial_L C_{\Omega_n}, \\ \dfrac{\partial v_n}{\partial \nu^\alpha} &= \lambda M_n^{q-p} v_n^q + v_n^p & \text{in } \Omega_n \times \{0\}. \end{cases} \tag{3.8}$$

By Arzelà-Ascoli Theorem (the solution is $\mathcal{C}^\gamma$, see Proposition 3.3), there exists a subsequence, which we denote again by v_n, which converges to some function v as $n \to \infty$. In order to see the problem satisfied by v we pass to the limit in the weak formulation of (3.8). We observe that $\|v_n\|_\infty \leq 1$ implies $\|v_n\|_{X_0^\alpha(C_\Omega)} \leq C$.

We define $d_n = \operatorname{dist}(x_n, \partial\Omega)$, then there are two possibilities as $n \to \infty$ according the behaviour of the ratio $\frac{d_n}{\mu_n}$:

1. $\left\{\dfrac{d_n}{\mu_n}\right\}_n$ is not bounded.

2. $\left\{\dfrac{d_n}{\mu_n}\right\}_n$ remains bounded.

In the first case, since $B_{d_n/\mu_n}(0) \subset \Omega_n$, and Ω_n is smooth, it is clear that Ω_n tends to $\mathbb{R}^N$ and v is a solution to

$$\left\{\begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla v) & = & 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \dfrac{\partial v}{\partial \nu^\alpha} & = & v^p & \text{on } \partial\mathbb{R}^{N+1}_+. \end{array}\right.$$

Moreover, $v(0,0) = 1$ and $v > 0$ which is a contradiction with Theorem 3.10.

In the second case, we may assume that $\frac{d_n}{\mu_n} \to s \geq 0$ as $n \to \infty$. As a consequence, passing to the limit, the domains Ω_n converge (up to a rotation) to some half-space $H_s = \{x \in \mathbb{R}^N : x_N > -s\}$. We obtain here that v is a solution to

$$\left\{\begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla v) & = & 0 & \text{in } H_s \times (0,\infty), \\ \dfrac{\partial v}{\partial \nu^\alpha} & = & v^p & \text{on } H_s \times \{0\}, \end{array}\right.$$

with $\|v\|_\infty = 1$, $v(0,0) = 1$. In the case $s = 0$ this is a contradiction with the continuity of v. If $s > 0$, the contradiction comes from Theorem 3.11. $\square$

We next prove a uniqueness result for solutions with small norm.

Theorem 3.12 *There exists at most one solution to problem (P_λ) with small norm.*

We follow closely the arguments in [4], so we establish the following previous result:

Lemma 3.13 *Let z be the unique solution to problem (3.6). There exists a constant $\beta > 0$ such that*

$$\|\phi\|^2_{X_0^\alpha(C_\Omega)} - q\int_\Omega z^{q-1}\phi^2\,dx \geq \beta\|\phi\|^2_{L^2(\Omega)}, \quad \forall \phi \in X_0^\alpha(C_\Omega). \tag{3.9}$$

Proof. We recall that z can be obtained by minimization

$$\min\left\{\frac{1}{2}\|\omega\|^2_{X_0^\alpha(C_\Omega)} - \frac{1}{q+1}\|w\|^{q+1}_{L^{q+1}(\Omega)} : \quad \omega \in X_0^\alpha(C_\Omega)\right\}.$$

As a consequence,

$$\|\phi\|^2_{X_0^\alpha(C_\Omega)} - q\int_\Omega z^{q-1}\phi^2\,dx \geq 0, \quad \forall \phi \in X_0^\alpha(C_\Omega).$$

This implies that the first eigenvalue a_1 of the linearized problem

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla\phi) &= 0, \qquad \text{in } C_\Omega, \\ \phi &= 0, \qquad \text{on } \partial_L C_\Omega, \\ \dfrac{\partial\phi}{\partial\nu^\alpha} - qz^{q-1}\phi &= a\phi, \qquad \text{on } \Omega\times\{0\}, \end{cases}$$

is nonnegative.

Assume first that $a_1 = 0$ and let φ be a corresponding eigenfunction. Taking into account that z is the solution to (3.6) we obtain that

$$q\int_\Omega z^q\varphi\,dx = \int_\Omega z^q\varphi\,dx$$

which is a contradiction.

Hence $a_1 > 0$, which proves (3.9). □

Proof of Theorem 3.12. Consider $A > 0$ such that $pA^{p-1} < \beta$, where β is given in (3.9). Now we prove that problem (P_λ) has at most one solution with L^∞-norm less than A.

Assume by contradiction that (P_λ) has a second solution $w = w_\lambda + v$ verifying $\|w\|_\infty < A$. Since w_λ is the minimal solution, it follows that $v > 0$ in $\Omega\times[0,\infty)$. We define now $\eta = \lambda^{\frac{1}{1-q}}z$, where z is the solution to (3.6). Then it verifies $-\div(y^{1-\alpha}\nabla\eta) = 0$, with boundary condition $\lambda\eta^q$. Moreover, w_λ is a supersolution to the problem that η verifies. Then by comparison, Lemma 3.2, applied with $f(t) = \lambda t^q$, $v = \eta$ and $w = w_\lambda$, we get

$$w_\lambda \geq \lambda^{\frac{1}{1-q}}z \qquad \text{on } \Omega\times\{0\}. \tag{3.10}$$

Since $w = w_\lambda + v$ is solution to (P_λ) we have, on $\Omega\times\{0\}$,

$$\frac{\partial(w_\lambda+v)}{\partial\nu^\alpha} = \lambda(w_\lambda+v)^q + (w_\lambda+v)^p \leq \lambda w_\lambda^q + \lambda q w_\lambda^{q-1}v + (w_\lambda+v)^p,$$

where the inequality is a consequence of the concavity, hence

$$\frac{\partial v}{\partial\nu^\alpha} \leq \lambda q w_\lambda^{q-1}v + (w_\lambda+v)^p - w_\lambda^p.$$

Moreover, (3.10) implies $w_\lambda^{q-1} \geq \lambda^{-1}z^{q-1}$. From the previous two inequalities we get

$$\frac{\partial v}{\partial\nu^\alpha} \leq qz^{q-1}v + (w_\lambda+v)^p - w_\lambda^p.$$

Using that $\|w_\lambda + v\|_\infty \leq A$, we obtain $(w_\lambda+v)^p - w_\lambda^p \leq pA^{p-1}v$. As a consequence,

$$\frac{\partial v}{\partial\nu^\alpha} - qz^{q-1}v \leq pA^{p-1}v.$$

Taking v as a test function and $\phi = v$ in (3.9) we arrive to

$$\beta \int_\Omega v^2\, dx \le pA^{p-1} \int_\Omega v^2\, dx.$$

Since $pA^{p-1} < \beta$ we conclude that $v \equiv 0$, which gives the desired contradiction. $\square$

Remark 3.3 This proof also provides the asymptotic behavior of w_λ near $\lambda = 0$, namely $w_\lambda \approx \lambda^{\frac{1}{1-q}} z$, where z is the unique solution to problem (3.6).

A Appendix: A trace inequality

A.1 Preliminaries. The problem in the whole space

Here we recall the definition of the fractional Laplacian in the whole space and its local representation by means of the extension to one more variable, as it is done in [17].

Let u be a regular function in $\mathbb{R}^N$. We say that $w = \mathrm{E}_\alpha(u)$ is its α-harmonic extension to the upper half-space, $\mathbb{R}^{N+1}_+$, if w is a solution to the problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}(y^{1-\alpha}\nabla w) & = & 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w & = & u & \text{on } \mathbb{R}^N \times \{y = 0\}. \end{array} \right.$$

In [17] it is proved that

$$\lim_{y\to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}(x,y) = -\kappa_\alpha (-\Delta)^{\alpha/2} u(x), \tag{A.1}$$

where $\kappa_\alpha = \frac{2^{1-\alpha}\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)}$.

The appropriate functional spaces to work with are $X^\alpha(\mathbb{R}^{N+1}_+)$ and $\dot{H}^{\alpha/2}(\mathbb{R}^N)$, defined as the completion of $\mathcal{C}_0^\infty(\overline{\mathbb{R}^{N+1}_+})$ and $\mathcal{C}_0^\infty(\mathbb{R}^N)$, respectively, under the norms

$$\begin{aligned} \|\phi\|^2_{X^\alpha} &= \int_{\mathbb{R}^{N+1}_+} y^{1-\alpha} |\nabla \phi(x,y)|^2\, dxdy, \\ \|\psi\|^2_{\dot{H}^{\alpha/2}} &= \int_{\mathbb{R}^N} |2\pi\xi|^\alpha |\widehat{\psi}(\xi)|^2\, d\xi = \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/4}\psi(x)|^2\, dx. \end{aligned}$$

The extension operator is well defined for smooth functions through a Poisson kernel, whose explicit expression is given in [17]. It can also be defined in the space $\dot{H}^{\alpha/2}(\mathbb{R}^N)$, and in fact

$$\| \,\mathrm{E}_\alpha(\psi)\|_{X^\alpha} = c_\alpha \|\psi\|_{\dot{H}^{\alpha/2}}\,, \quad \forall\, \psi \in \dot{H}^{\alpha/2}(\mathbb{R}^N), \tag{A.2}$$

where $c_\alpha = \sqrt{\kappa_\alpha}$, see Lemma A.2. On the other hand, for a function $\phi \in X^\alpha(\mathbb{R}^{N+1}_+)$, we will denote its trace on $\mathbb{R}^N \times \{y = 0\}$ as $\mathrm{Tr}(\phi)$. This trace operator is also well defined and it satisfies

$$\| \,\mathrm{Tr}(\phi)\|_{\dot{H}^{\alpha/2}} \le c_\alpha^{-1} \|\phi\|_{X^\alpha}. \tag{A.3}$$

A.2 The trace immersion

From (A.3), the Sobolev embedding yields then that the trace also belongs to $L^{2^*_\alpha}(\mathbb{R}^N)$, where $2^*_\alpha = \frac{2N}{N-\alpha}$. Even the best constant associated to this inclusion is attained and can be characterized. Although most of the results used in order to prove the following theorem are known, we have collected them for the readers convenience.

Theorem A.1 *For every $z \in X^\alpha(\mathbb{R}^{N+1}_+)$ it holds*

$$\left(\int_{\mathbb{R}^N} |v(x)|^{\frac{2N}{N-\alpha}}\, dx \right)^{\frac{N-\alpha}{N}} \le S(\alpha, N) \int_{\mathbb{R}^{N+1}_+} y^{1-\alpha} |\nabla z(x,y)|^2\, dxdy, \tag{A.4}$$

where $v = \mathrm{Tr}(z)$. The best constant takes the exact value

$$S(\alpha, N) = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{N-\alpha}{2})(\Gamma(N))^{\frac{\alpha}{N}}}{2\pi^{\frac{\alpha}{2}}\Gamma(\frac{2-\alpha}{2})\Gamma(\frac{N+\alpha}{2})(\Gamma(\frac{N}{2}))^{\frac{\alpha}{N}}} \tag{A.5}$$

and it is achieved when v takes the form

$$v(x) = \tau^{\frac{N-\alpha}{2}} \left(|x - x_0|^2 + \tau^2 \right)^{-\frac{N-\alpha}{2}}, \tag{A.6}$$

for some $x_0 \in \mathbb{R}^N$, $\tau > 0$, and $z = \mathrm{E}_\alpha(v)$.

The analogous results for the classical Laplace operator can be found in [23, 33].

Lemma A.2 *Let $v \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$ and let $z = \mathrm{E}_\beta(v)$ be its β-harmonic extension, $\beta \in (\alpha/2, 2)$. Then $z \in X^\alpha(\mathbb{R}^{N+1}_+)$ and moreover there exists a positive universal constant $c(\alpha, \beta)$ such that*

$$\|v\|_{\dot{H}^{\alpha/2}} = c(\alpha, \beta) \|z\|_{X^\alpha}. \tag{A.7}$$

In particular if $\beta = \alpha$ we have $c(\alpha, \alpha) = 1/\sqrt{\kappa_\alpha}$.

Inequality (A.4) needs only the case $\beta = \alpha$, which is deduced directly from the proof of the local characterization of $(-\Delta)^{\alpha/2}$ in [17]. The calculations performed in [17] can be extended to cover the range $\alpha/2 < \beta < 2$ and in particular includes the case $\beta = 1$ proved in [42].

Proof. Since $z = \mathrm{E}_\beta(v)$, by definition z solves $\operatorname{div}(y^{1-\beta}\nabla z) = 0$, which is equivalent to

$$\Delta_x z + \frac{1-\beta}{y}\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$$

Taking Fourier transform in $x \in \mathbb{R}^N$ for $y > 0$ fixed, we have

$$-4\pi^2|\xi|^2\hat{z} + \frac{1-\beta}{y}\frac{\partial \hat{z}}{\partial y} + \frac{\partial^2 \hat{z}}{\partial y^2} = 0.$$

and $\hat{z}(\xi, 0) = \hat{v}(\xi)$. Therefore $\hat{z}(\xi, y) = \hat{v}(\xi)\phi_\beta(2\pi|\xi|y)$, where ϕ_β solves the problem

$$-\phi + \frac{1-\beta}{s}\phi' + \phi'' = 0, \qquad \phi(0) = 1, \quad \lim_{s\to\infty}\phi(s) = 0. \tag{A.8}$$

In fact, ϕ_β minimizes the functional

$$H_\beta(\phi) = \int_0^\infty (|\phi(s)|^2 + |\phi'(s)|^2)s^{1-\beta}\,ds.$$

and it can be shown that it is a combination of Bessel functions, see [31]. More precisely, ϕ_β satisfies the following asymptotic behaviour

$$\begin{aligned} \phi_\beta(s) &\sim 1 - c_1 s^\beta, &\quad& \text{for } \ s \to 0, \\ \phi_\beta(s) &\sim c_2 s^{\frac{\beta-1}{2}}e^{-s}, && \text{for } \ s \to \infty, \end{aligned} \tag{A.9}$$

where

$$c_1(\beta) = \frac{2^{1-\beta}\Gamma(1-\frac{\beta}{2})}{\beta\Gamma(\frac{\beta}{2})}, \quad c_2(\beta) = \frac{2^{\frac{1-\beta}{2}}\pi^{1/2}}{\Gamma(\frac{\beta}{2})}.$$

Now we observe that

$$\begin{aligned} \int_{\mathbb{R}^N}|\nabla z(x,y)|^2\,dx &= \int_{\mathbb{R}^N}\left(|\nabla_x z(x,y)|^2 + \left|\frac{\partial z}{\partial y}(x,y)\right|^2\right)dx \\ &= \int_{\mathbb{R}^N}\left(4\pi^2|\xi|^2|\hat{z}(\xi,y)|^2 + \left|\frac{\partial \hat{z}}{\partial y}(\xi,y)\right|^2\right)d\xi. \end{aligned}$$

Then, multiplying by $y^{1-\alpha}$ and integrating in y,

$$\begin{aligned} &\int_0^\infty\int_{\mathbb{R}^N} y^{1-\alpha}|\nabla z(x,y)|^2\,dxdy \\ &= \int_0^\infty\int_{\mathbb{R}^N} 4\pi^2|\xi|^2|\hat{v}(\xi)|^2\big(|\phi_\beta(2\pi|\xi|y)|^2 + |\phi_\beta'(2\pi|\xi|y)|^2\big)y^{1-\alpha}\,d\xi dy \\ &= \int_0^\infty (|\phi_\beta(s)|^2 + |\phi_\beta'(s)|^2)s^{1-\alpha}\,ds \int_{\mathbb{R}^N}|2\pi\xi|^\alpha|\hat{v}(\xi)|^2\,d\xi. \end{aligned}$$

Using (A.9) we see that the integral $\int_0^\infty (|\phi_\beta|^2 + |\phi'_\beta|^2) s^{1-\alpha}\, ds$ is convergent provided $\beta > \alpha/2$. This proves (A.7) with $c(\alpha,\beta) = (H_\alpha(\phi_\beta))^{-1/2}$. $\square$

Remark A.1 If $\beta = 1$ we have $\phi_1(s) = e^{-s}$, and $H_\alpha(\phi_1) = 2^{\alpha-1}\Gamma(2-\alpha)$, see [42]. Moreover, when $\beta = \alpha$, integrating by parts and using the equation in (A.8), and (A.9), we obtain

$$H_\alpha(\phi_\alpha) = \int_0^\infty [\phi_\alpha^2(s) + (\phi'_\alpha)^2(s)] s^{1-\alpha}\, ds = -\lim_{s\to 0} s^{1-\alpha}\phi'_\alpha(s) = \alpha c_1(\alpha) = \kappa_\alpha. \tag{A.10}$$

Lemma A.3 *Let $z \in X^\alpha(\mathbb{R}^{N+1}_+)$ and let $w = \mathrm{E}_\alpha(\mathrm{Tr}(z))$ be its α-harmonic associated function (the extension of the trace). Then*

$$\|z\|^2_{X^\alpha} = \|w\|^2_{X^\alpha} + \|z-w\|^2_{X^\alpha}.$$

Proof. Observe that, for $h = z - w$, we have

$$\|z\|^2_{X^\alpha} = \int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}(|\nabla w|^2 + |\nabla h|^2 + 2\langle \nabla w, \nabla h\rangle).$$

But, since $\mathrm{Tr}(h) = 0$, we have $\displaystyle\int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}\langle \nabla w, \nabla h\rangle\, dxdy = 0.$ $\square$

Lemma A.4 *If $g \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, and $f \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$, then there exists a constant $\ell(\alpha, N) > 0$ such that*

$$\left|\int f(x)\, g(x)\, dx\right| \le \ell(\alpha,N)\|f\|_{\dot{H}^{\alpha/2}}\|g\|_{\frac{2N}{N+\alpha}}. \tag{A.11}$$

Moreover, the equality in (A.11) *with the best constant holds when f and g take the form* (A.6)*.*

The proof follows by an standard argument that can be found, for instance in [22, 42].

Proof. By Parçeval's identity and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N} f(x)\, g(x)\, dx\right)^2 &= \left(\int_{\mathbb{R}^N} \widehat{f}(\xi)\, \widehat{g}(\xi)\, d\xi\right)^2 \\ &\le \left(\int_{\mathbb{R}^N} |2\pi\xi|^\alpha\, |\widehat{f}(\xi)|^2\, d\xi\right)\left(\int_{\mathbb{R}^N} |2\pi\xi|^{-\alpha}\, |\widehat{g}(\xi)|^2\, d\xi\right).\end{aligned}$$

The second term can be written using [32] as

$$\int_{\mathbb{R}^N} |2\pi\xi|^{-\alpha} |\widehat{g}(\xi)|^2 \, d\xi = b(\alpha, N) \int_{\mathbb{R}^{2N}} \frac{g(x)g(x')}{|x - x'|^{N-\alpha}} \, dx dx',$$

where

$$b(\alpha, N) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{N/2} \Gamma(\frac{\alpha}{2})},$$

We now use the following Hardy-Littlewood-Sobolev inequality, see again [32],

$$\int_{\mathbb{R}^{2N}} \frac{g(x)g(x')}{|x - x'|^{N-\alpha}} \, dx dx' \le d(\alpha, N) \|g\|^2_{\frac{2N}{N+\alpha}},$$

where

$$d(\alpha, N) = \frac{\pi^{\frac{N-\alpha}{2}} \Gamma(\frac{\alpha}{2}) (\Gamma(N))^{\frac{\alpha}{N}}}{\Gamma(\frac{N+\alpha}{2}) (\Gamma(\frac{N}{2}))^{\frac{\alpha}{N}}},$$

with equality if g takes the form (A.6). From this we obtain the desired estimate (A.11) with the constant $\ell(\alpha, N) = \sqrt{b(\alpha, N) d(\alpha, N)}$.

When applying Cauchy-Schwarz's inequality, we obtain an identity provided the functions $|\xi|^{\alpha/2} \widehat{f}(\xi)$ and $|\xi|^{-\alpha/2} \widehat{g}(\xi)$ are proportional. This means

$$\widehat{g}(\xi) = c|\xi|^\alpha \widehat{f}(\xi) = c[(-\Delta)^{\alpha/2} f]^\wedge (\xi).$$

We end by observing that if g takes the form (A.6) and $g = c(-\Delta)^{\alpha/2} f$ then f also takes the form (A.6). □

Proof of Theorem A.1. We apply Lemma A.4 with $g = |f|^{\frac{N+\alpha}{N-\alpha} - 1} f$, then use Lemma A.2 and conclude using Lemma A.3. The best constant is $S(\alpha, N) = \ell^2(\alpha, N)/\kappa_\alpha$. □

Related to this result we quote the work [19], where it is proved that the only positive regular solutions to $(-\Delta)^{\alpha/2} f = c f^{\frac{N+\alpha}{N-\alpha}}$ take the form (A.6).

Remark A.2 If we let α tend to 2, when $N > 2$, we recover the classical Sobolev inequality for a function in $H^1(\mathbb{R}^N)$, with the same constant, see [40]. In order to pass to the limit in the right-hand side of (A.4), at least formally, we observe that $(2 - \alpha) y^{1-\alpha} \, dy$ is a measure on compact sets of $\mathbb{R}_+$ converging (in the weak-* sense) to a Dirac delta. Hence

$$\lim_{\alpha \to 2^-} \int_0^1 \left(\int_{\mathbb{R}^N} |\nabla z(x, y)|^2 \, dx \right) (2 - \alpha) y^{1-\alpha} \, dy = \int_{\mathbb{R}^N} |\nabla v(x)|^2 \, dx.$$

We then obtain

$$\left(\int_{\mathbb{R}^N} |v(x)|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \le S(N) \int_{\mathbb{R}^N} |\nabla v(x)|^2 \, dx,$$

with the best constant $S(N) = \lim_{\alpha\to 2^-} \frac{S(\alpha,N)}{2-\alpha} = \frac{1}{\pi N(N-2)}\left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}\right)^{\frac{2}{N}}$. It is achieved when v takes the form (A.6) with α replaced by 2.

Let now $v \in X_0^\alpha(C_\Omega)$. Its extension by zero outside the cylinder C_Ω can be approximated by functions with compact support in $\overline{\mathbb{R}^{N+1}_+}$. Thus, the trace inequality (A.4), together with Hölder's inequality, gives a trace inequality for bounded domains.

Theorem A.5 *For any $1 \le r \le \frac{2N}{N-\alpha}$, and every $z \in X_0^\alpha(C_\Omega)$, it holds*

$$\left(\int_\Omega |v(x)|^r\,dx\right)^{2/r} \le C(r,\alpha,N,|\Omega|)\int_{C_\Omega} y^{1-\alpha}|\nabla z(x,y)|^2\,dxdy, \qquad \text{(A.12)}$$

where $v = \mathrm{Tr}(z)$.

Acknowledgments.
CB and AdP partially supported by Spanish Project MTM2008-06326-C02-02; EC partially supported by Spanish Project MTM2009-10878.

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